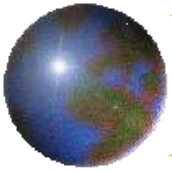


Multi-Period Planning

An Optimization Oriented Guide

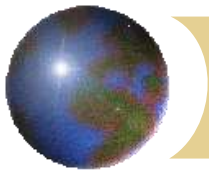
April 2021
Linus Schrage

An updated version of a presentation at
INFORMS Phoenix by Monday, 5 November 2018.
Also see INFORMS TutORials in OR 2018.
Keywords: Multi-period planning, Long range planning



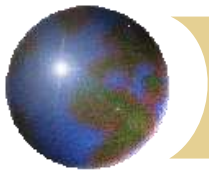
Multi-Period Optimization Problem, Typical Applications

- * Production Planning/Supply Chains :
What to produce/ship where & when. (Period = Day, or Week, Month, Year).
- * Financial models: What to invest in when, in face of uncertainty
- * Electric power generation/Unit Commitment, especially for hydro:
Startup, Shutdown costs, Ramp-up effects. (Hour).
Nonlinear interaction between water level * flow rate = power output. (Hour).
- * Natural gas distribution:
Nonlinear inventory storage because of compressibility.
- * Resource extraction / Oil fields, Mines:
Cost and Rate of production depend upon amount already extracted.
- * Staff scheduling, - hospitals, etc.
Complicated rules for time off, mix of skills available each period.
- * Routing over time,
FTL vs. LTL, Time windows.
- * Population models in Forests, Wild life management, Cheese/Wine/aged products:
Tradeoffs between harvest rate and population growth.



Principal question:

How to design and use
optimization based multi-period planning models.



Outline:

History of Multi-Period Planning + Optimization.

Backlogging, Lost Sales, the Basic Inventory Equation.

Objectives: NPV and more.

Multiple Dimensions of Modeling, Parkinson's Law of Modeling.

Rolling/Sliding Schedules and Combatting Nervousness.

Inventory Types: Attrition, Perishable, Growing, FIFO vs. LIFO, Compressible.

Planning Horizon Length, How to Choose.

Period Length, Choosing.

Start and End Conditions.

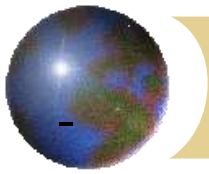
Steady State and Steady Growth Rate Models, and Insight Provided.

Resource, Precedence, & Changeover Constraints: How Not to Represent.

Presenting Results of Multi-period Plans, Space-time Diagrams.

Taxes, how to represent.

Uncertainty: How to represent compactly.



History of Multi-Period Planning & Optimization

Dates at least to biblical times.

(According to G. Dantzig) The parable of Joseph interpreting the Pharaoh's dream about the seven fat cows and the seven lean cows.

Joseph: Seven good years will be followed by seven years that are lean.

Planning advice:

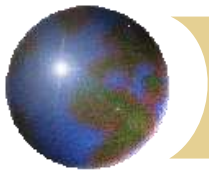
Do multi-period planning.

Build up inventories of grain during fat years.

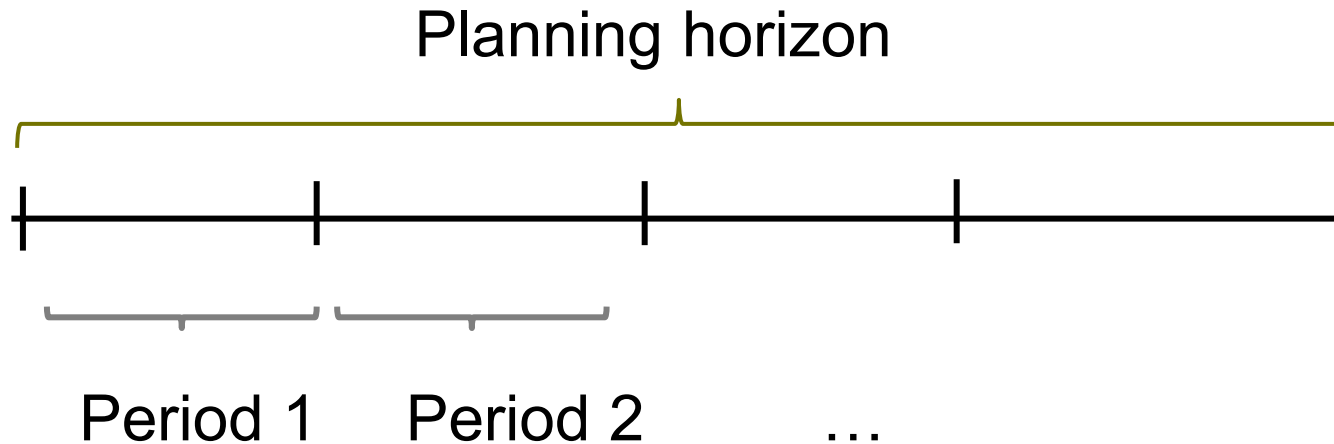
Pharaoh: That worked out great!

Joseph, how did you know what approach to use?

Joseph: _____
Use lean year programming

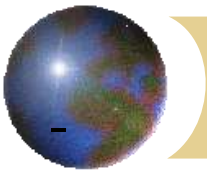


Some Basic Ideas



What should be the length of:
Planning horizon, Period?

What should be the beginning condition? Ending condition?



History of Multi-Period Planning & Optimization, More Recently:

Holt, Modigliani, Muth, & Simon Planning Model (as Quadratic Program)

Done for a Pittsburgh Paint Company, 1955

Given demand forecasts for the next 12 months,
for a single (aggregated) product,
we have a single work force,

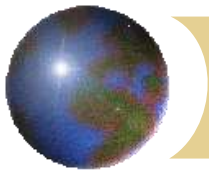
What are our options for satisfying the (predictably) fluctuating demand?

Decisions: Change work force level from period to period?

Use overtime in some periods?

Carry inventory, or back log demand?

Hanssmann & Hess 1960: gave a more
practical reformulation as an LP.



The Hanssmann-Hess Model of the Paint Company

(see: HanssmannHess.lng, HanssmannHessx.xlsx)

Ending inventory = beginning inventory + production – demand;

$$I_t = I_{t-1} + P_t - d_t; \quad (\text{Note all variables } \geq 0.)$$

Workforce level at end of period = beginning level + hiring – firing;

$$W_t = W_{t-1} + H_t - F_t;$$

Cannot produce more than allowed by regular time + overtime;

$$P_t \leq k*(W_t + O_t);$$

Positive inventory – negative inventory = inventory;

$$IP_t - BKLOG_t = I_t;$$

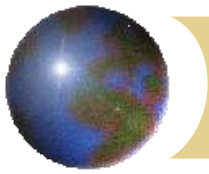
$$P_t, W_t, H_t, F_t, O_t, IP_t, BKLOG_t \geq \underline{0};$$

Minimize costs of

Regular time + Overtime + Hiring + Firing + Inventory + Backlogging;

Minimize

$$\sum_t (c_r * W_t + c_o * O_t + c_h * H_t + c_f * F_t + c_c * IP_t + c_b * BKLOG_t)$$



The Hanssmann-Hess Model, Backlogging vs. Lost Sales

Two extreme ways a customer can react when we cannot supply the order:

Backlog:

I'll wait, send material as soon as you can produce
(and give me a discount?).

Lost Sales:

I cannot wait. Cancel my order. I will get what I need elsewhere.

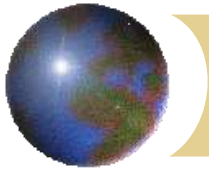
The inventory balance equations for the lost sales case are:

Ending inventory = beginning inventory + production – sales,

$$I_t = I_{t-1} + P_t - SALES_t; \quad (\text{Note all variables } \geq 0.)$$

$$SALES_t + LOSTSALES_t = d_t;$$

Q: Suppose fraction f of unsatisfied demand is lost, rest backlogged?



Objectives: NPV and more

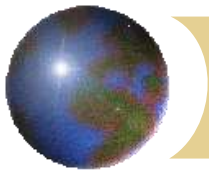
If C_t = cash throw off in period t , and r is given interest rate/cost of capital, a typical objective is to maximize the Net Present Value (NPV):

$$\text{Maximize } C_0 + C_1/(1+r) + C_2/(1+r)^2 + \dots + C_n/(1+r)^n$$

Sometimes the Internal Rate of Return (IRR) is also used as a metric, i.e., find an r that satisfies:

$$0 = C_0 + C_1/(1+r) + C_2/(1+r)^2 + \dots + C_n/(1+r)^n$$

High IRR is preferred to a low IRR.



Objectives: NPV and more, . .

Admonitions regarding IRR:

- a) IRR should not be used if a project has future negative cash flows.
- b) IRR should not be used for choosing between alternatives.

To illustrate (a), consider the cash flow (we must clean up at project end):

Year:	0	1	2
Cash:	-1	2.5	-1.55

This project has two internal rates of return: 13.82% and 36.18%.

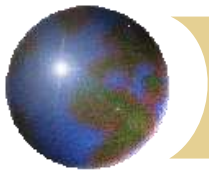
Is this an attractive project?

NPV analysis will (correctly)

reject if our cost of capital is $< 13.8\%$ per year,

accept if our cost of capital is $= 24\%$ per year, and

reject if our cost of capital is $> 37\%$ per year.



Objectives: NPV and more,

Admonitions regarding IRR, to illustrate (b),
suppose we can undertake either project X or project Y, but not both.
E.g., alternative ways to exploit a specific potential oil field.

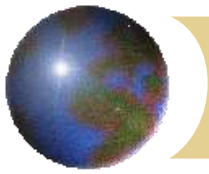
Year:	<u>0</u>	<u>1</u>		<u>IRR</u>	<u>NPV @ 15%</u>
X:	-1	1.2	;	20%	0.0435
Y:	-100	118	;	18%	2.6087

Project X has an IRR of 20% and project Y has an IRR of 18%.

So should we choose X?

Now supply the additional information that our cost of capital is 15%.

NPV says choose Y rather than X.



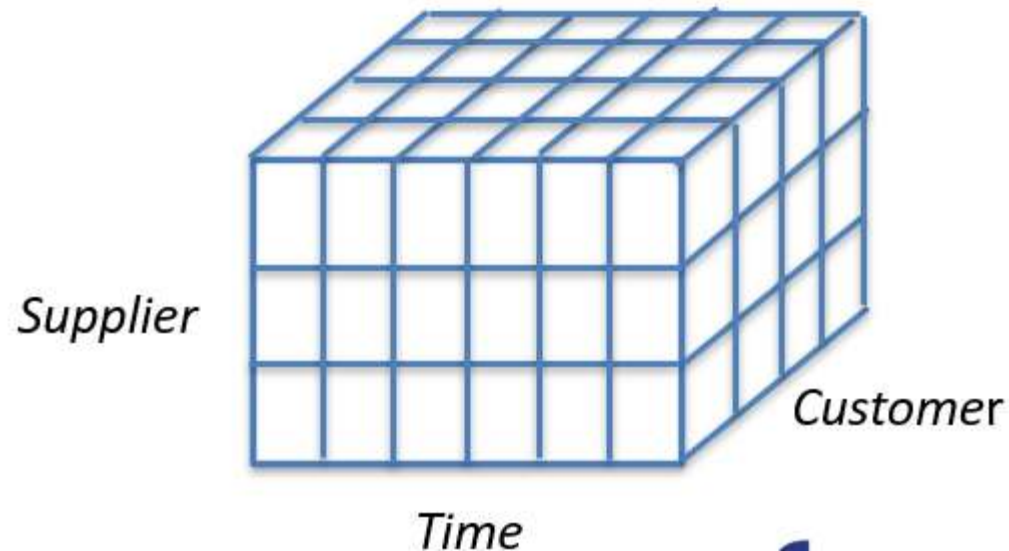
Multiple Dimensions of Modeling, Parkinson's Law of Modeling:

User wants to add more and more details until
time to solve an N period model approaches N periods.

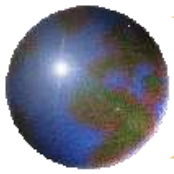
Ex.: We can compute very accurate 5-day-ahead weather forecasts, but...
it takes 6 days to do the computation.

Dimensions of a planning model:

- Suppliers,
- Customers,
- Time periods,
- Products,
- Process types,
- Transport modes, (rail, etc.),
- Scenarios,
- Resources, e.g., labor skills.



Which dimensions should be included,
and how many slices of each?



Multi-Period Optimization Models, General Structure

Each period is a copy of various single period models,
(blending, product mix, etc.) tied together by introducing:

1) Inventory variables:

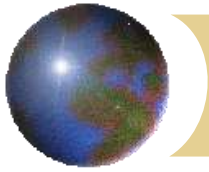
$Inv(c, t)$ = inventory of commodity c at end of period t ,

2) A “material balance” or “sources = uses” constraint
for each commodity and period:

$$Inv(c, t-1) + Production(c, t) = Sales(c, t) + Inv(c, t);$$

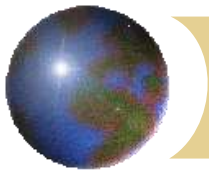
or in difference form:

$$Inv(c, t) - Inv(c, t-1) = Production(c, t) - Ship_out(c, t);$$



Multi-Period Optimization Models, Complications & Issues

- * Rolling/Sliding Schedules and Nervousness.
- * Types of inventories: attrition, growth, FIFO vs. LIFO, perishable, compressible.
- * Planning horizon length
- * End effects, Choosing planning horizon,
- * Steady state and steady growth rate models, and the insight they provide.
- * Resource, precedence, & changeover constraints: a simple bad way, a better way.
- * How to present results of multi-period plans, space-time diagrams.
- * Taxes, how to represent.
- * Uncertainty: How to represent compactly.



Multi-Period Optimization Models, Common Features/Problem

Rolling/Sliding Schedules

Results from model are used in rolling/sliding fashion:

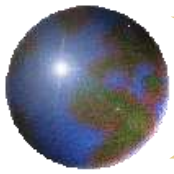
Loop:

- 1) Solve an N period model,
- 2) Implement the results from the first one or two periods of solution;
- 3) Update forecasts, Slide the model forward one period,

Repeat

Problem:

Model nervousness: As forecasts change, the recommended solution may change a lot. The folks “on the production floor” may be unhappy with the significant change in plans, period to period. Suppliers like to know your long run production plans. They will be unhappy if your plans change dramatically with each plan release.



Nervousness and Rolling/Sliding Schedules

How to combat solution nervousness when using rolling or sliding schedules.

Observation: For typical real planning problems, there are many (close to) optima.

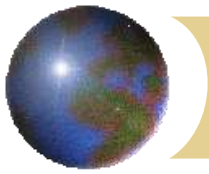
Approach used successfully in scheduling ships, plant closings/openings, production of breakfast cereal, by Brown, Dell, and Wood (1997), is to

Specify a “reference” solution (e.g., the solution from the run of previous month).
Define secondary objective of minimize deviation of the current solution from reference.

If zero weight on the secondary objective \implies theoretically optimal solution.

If high weight on the secondary objective, \implies reference solution returned.

If a modest weight on secondary objective, \implies alternate (almost) optimum close to reference solution.



Choosing a Planning Horizon

Simple Suggestion:

If solving an $N+1$ period problem and an N period problem give the same first period solution, then use N as the planning horizon.

This may be too long or too short.

Too long:

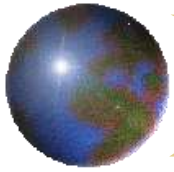
Alternative optima,

or close to alternative in which first period decisions are not the same.

Too short:

Adding November made no difference in the January (period 1) solution, but when you added December, there was a big change in January solution.

Note, there is a substantial literature on identifying planning horizons for moderately simple multi-period lotsizing problems using above rule.



Choosing a Planning Horizon

General approach:

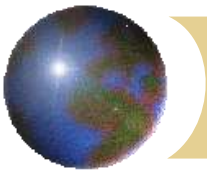
N is a good planning horizon length if

- 1) Solve an N period problem;
- 2) Solve the two problems:
 - A) For the decisions in the first period fixed, solve an $N+K$ period problem,
 - B) For the decisions in the first period not fixed, solve $N+K$ period problem,

If the costs of the solutions to A and B are not significantly different, then N is a reasonable planning horizon.

Variation:

If requiring integrality in first $N+K$ periods, gives the same first period solution as requiring integrality in first N periods, then OK to require integrality in only first N periods.



Choosing a Planning Horizon: Best Time to Harvest Problems

When to harvest/sell some of assets that grow in value over time?

E.g:

Christmas trees:

harvest at 7 to 12 years,

Timber for paper or building:

harvest at 25 to 40 years,

Wine, cheese,

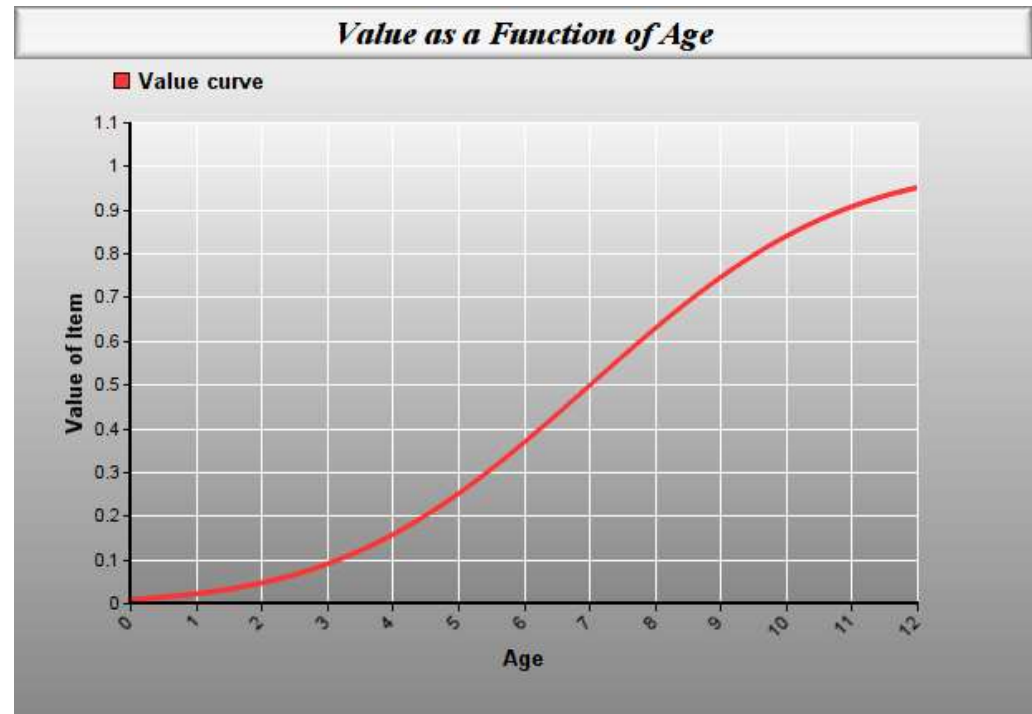
Planning horizon should be

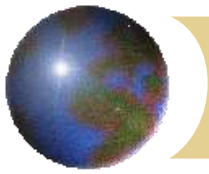
\geq low multiple of

typical time-to-harvest, e.g.,

≥ 20 for a Christmas tree farm,

≥ 60 years for a forest plot.





Choosing a Period Length

Examples:

In scheduling daily work patterns at a telephone call center, distribution centers, etc., breaks are a multiple of 15 minutes, so typical period length is 15 minutes.

In scheduling nurses, a traditional period is a week.

In scheduling electrical generators, it may take an hour to bring a generator up to full power, so a period might be an hour.

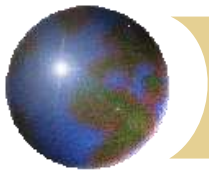
Period length need not be constant, e.g.,

Periods: 1-4 = 1 quarter/3 months,

Periods: 5-6 = 1 year,

Periods: 6-7 = 2 years,

Period 8 = 1 year repeated forever.



Choosing a Period Length

A Mathematical approach:

Given the lengths various activities, use the (approximate) Greatest Common Divisor (GCD).

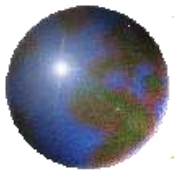
Example:

Assigning aircraft to multiple runways.

Suppose required landing separations (secs) at an airport are:

<u>Previous</u>	<u>Following</u>	Light	Medium	Heavy
Light		45	60	75
Medium		90	75	75
Heavy		105	75	75

The GCD is 15 seconds, so a plausible period length is 15 seconds for a discrete time model.



Choosing Number Periods For an Activity

Suppose we chose a period length is 0.5 hours, and a certain activity has a forecasted duration = 1.22 hours, what should we use as the activity duration in periods?

Rounding to minimize relative error.

t = actual activity time,

e.g. 1.22 hrs,

P = period length,

e.g., 0.5 hrs

Define:

$I = \text{FLOOR}(t/P)$ = periods for activity rounded down, e.g., 2,

$f = t/P - I$ = fractional part = error if rounded down, e.g., 0.44

td = activity time in discrete periods (either I or $I + 1$),

We minimize relative error of rounded value by the rule:

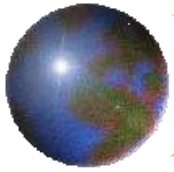
If $f < I / (2 * I + 1)$,

e.g. $0.44 < 2 / 5$?

then $td = I$, else $td = I + 1$.

E.g., 2.44 would get rounded to 3, 2.35 would get rounded to 2.

Notice, td is always ≥ 1 .



Period Length: Let it Go to 0?

Why not let period length go to 0?

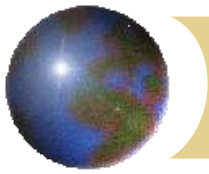
- + more precision
- computationally expensive, # periods \rightarrow infinity.

Two cases where continuous time might be OK.

- 1) For simple systems, may reduce to a problem in Ordinary Differential Equations (ODE).
Usually results in $\exp(t)$ terms.
- 2) Finite number of decision points E.g.,
routing of aircraft at specified departure points.

Example of (1):

Continuous compounding, partition each year into n periods,
 $(1 + r)^t \rightarrow (1 + r/n)^{(n*t)} \rightarrow \exp(r*t)$.



Period Length $\rightarrow 0$, the Bass Marketing Model

Two populations:

1) Candidates to adopt, 2) Adopters (already bought)

Parameters (to be estimated):

M = estimate of final market size,

= Candidates + Adopters (at any instant),

= the number of customers that will eventually buy.

P = probability that any candidate customer in a given period will adopt/purchase the product just by chance,

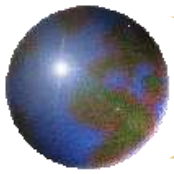
Q = rate at which any candidate customer is induced to buy by existing customers. The greater the number existing customers, the greater the inducement.

These folk are the followers;

Compute/Predict:

$S(t) =$ sales in period t .

$CUM(t) = CUM(t-1) + S(t)$;



Period Length $\rightarrow 0$, the Bass Marketing Model, cont.

The computational model:

$S(t)$ = sales in period t ,

= Prob(any candidate customer will buy in t) *

CandidateMarketSize;

= ($P + Q * CUM(t - 1) / M$) * ($M - CUM(t - 1)$);

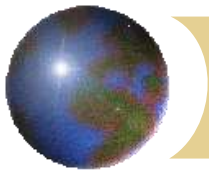
Limit in continuous time

$$CUM(t) = M * (1 - G(t)) / (1 + G(t) * Q/P)$$

where $G(t) = \exp(-(P+Q)*t)$

Note: $G(0) = 1$, goes to 0.

Gives an "S" shaped curve for $CUM(t)$.

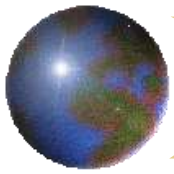


Period Length $\rightarrow 0$, Other examples.

- a) SIR Epidemic model. (Satsuma et al.)
Generalization of Bass. Three populations:
Susceptible \rightarrow Infected \rightarrow Recovered,

- b) Time-to-climb-to-cruise altitude for aircraft. (Cots et al.)
For each instant/period of climb plan horizon:
choose thrust and angle of attack to strike best
compromise over total plan horizon between fuel
consumption and distance traveled to destination.

- c) Dynamic soaring by gliders exploiting wind shear.



Multi-Period Optimization Models, Steady State Solutions

Steady state solutions are sometimes of interest. Why?

Easier to understand.

Users like them, “There is a flight to Tucson every Tuesday.”

Easier to solve.

May give “Insight” for long range planning, e.g., of capacities.

Useful as the final period of a multi-period model.

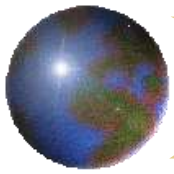
Finding a Steady State Constant solution is conceptually simple:

Make the ending conditions = beginning conditions,
e.g., the inventory levels.

A slightly more general definition of steady state or stationarity is if the growth rate remains unchanged from one period to the next.

Slightly more precisely, there is a scalar, λ , so that for every inventory c :

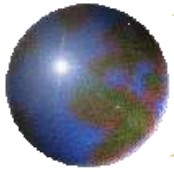
$$INV(c, t) = \lambda * INV(c, t-1); \text{ Simple steady state has } \lambda = 1;$$



Suppose we require that the ending conditions = beginning conditions. This may result in alternative optima, which can increase solve time if solving an integer program.

Example: We want find an optimal food menu for a 4 week period. There are various constraints such as nutritional requirements each day and variety constraints among days, e.g., cannot serve the same meal two days in a row. If there are no day specific constraints, e.g., must serve fish on Friday, then there will be $4 \times 7 = 28$ alternative optima.

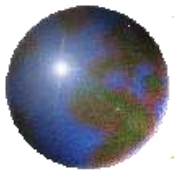
Another solution obtained from the previous by shifting the solution forward and around one day.



Steady State Solutions: Cyclic Solutions, Choosing a Cycle Length

Simplest form of a steady state is to add a constraint that the end of the period state must equal the conditions at the beginning of the period.

Sometimes is useful to allow the cycle length to be more than one period. This is particularly true in routing or staffing problems, e.g., you might require that the solution repeat every four weeks rather than every week.



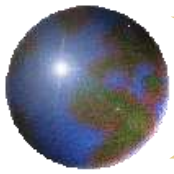
Population Models, Steady State Solutions

A slightly more general definition of steady state or stationarity is if the growth rate remains unchanged from one period to the next.

Consider a multi-period population model where:

$P(s,t)$ = population size of species s , in period t ,
where different species might be:

- 1 month-old cheese, 2-month old cheese,
- 2-month old rabbits, 3-month old rabbits,
- 4-month old foxes,
- 1-year old pine trees, 2-year old pine trees, etc.



Population Models, Steady State, Eigenvalues

A linear model would be represented by a matrix A describing how the different species interact, so in vector notation:

$$P(t) = A * P(t-1);$$

If we are interested in steady (exponential) growth or decay, we might ask, is there a scalar constant growth rate, λ , so that:

$$P(t) = A * P(t-1) = \lambda * P(t-1);$$

i.e., each species grows or decays by the same factor λ each period, or more simply, is there a “steady state growth” solution to:

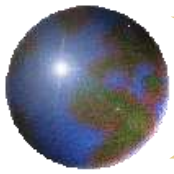
$$A * P = \lambda * P;$$

In general, there are multiple eigenvalues λ , some < 1 , some > 1 , and eigenvectors, P .

This is the eigenvalue equation.

This is easy to solve in LINGO with its matrix commands, e.g.,

```
LAMBDA R, VR, LAMBDA I, VI, err = @EIGEN( A );
```

Population and Epidemiology Models

Track how sizes of various population segments change over time.

Example: An epidemic. Track three segments:

Susceptible, Infected, Recovered, the so-called SIR model.

$$N = S(t) + I(t) + R(t),$$

total population size,

$$S(t+1) = S(t) - \beta * S(t) * I(t) / N,$$

Susceptible in period $t+1$,

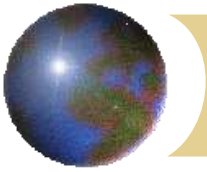
β = infection rate,

$$I(t+1) = I(t) + \beta * S(t) * I(t) / N - \gamma * I(t),$$

where γ = recovery rate,

$$R(t+1) = R(t) + \gamma * I(t);$$

An important ratio is β/γ . If $\beta/\gamma > 1$, infections tend to increase.



Multi-Period Planning, How to Choose Boundary Conditions?

Ending conditions:

If we arbitrarily terminate our planning model at year five in the future, then an optimal solution to our model may, in reality, be an optimal solution to how to go out of business in five years.

Some of the options for handling the end effect are:

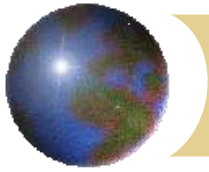
a) Truncation (default). Simply drop from the model all periods beyond N .

b) Primal limits. Place reasonable limits on inventories, etc. at the end of the final period.

E.g., in Pittsburgh Paint company, ending inventory was given a lower bound.

c) Salvage values/ dual prices. Place reasonable salvage values on final inventories, etc.

If you are an airline with fuel inventories, you may place a salvage value on fuel ending inventory.



Multi-Period Planning, How to Choose End Conditions?

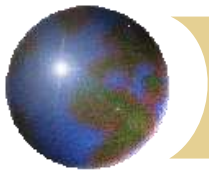
d) Infinite number of steady state periods.

Final period of the model represents an infinite number of periods for which the same decision applies in every period to infinity.

Net present value discounting is used in the objective function to make the final period comparable to the earlier finite periods.

This approach used by:

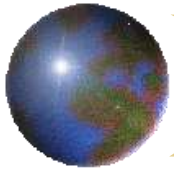
Carino et al. in their model of the Yasuda Kasai Insurance Company,
Peiser and Andrus in their model of Texas real estate development, and
Eppen, Martin, and Schrage in model of General Motors production planning.



Multi-Period Planning, How to Choose Boundary Conditions?

Beginning conditions:

- a) Use current state of real system.
- b) Start however you wish, but ending state must approximately match beginning state, e.g., if trying to model steady state behavior.
- c) “Irregular Operations” (IROPS) is a standard term in airlines for a system for how to best recover after a disruption, typically bad weather.
Problem: Given current state with resources (planes, crews, passengers) in the wrong place, what is the most efficient way of getting back on schedule?
Priorities, decisions, costs: Get stranded passengers to their destinations, which flights to cancel, which planes and crews to fly from stranded location to where to get back on schedule by, say, Monday.



Modeling Taxes

Essential tax computation equations are:

$$Profit_t - Loss_t = Revenue_t - Expense_t ;$$

May need to enforce:

$$\text{Either } Profit_t \text{ or } Loss_t = 0 ;$$

Note, these are taxable revenues and expenses.

$$UsesCash_t = SourcesCash_t ;$$

$$UsesCash_t = TaxRate_t * Profit_t + etc.$$

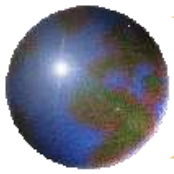
$$Expense_t = Depreciation_t + etc.$$

Complications:

Depreciation, LIFO vs. FIFO,

Loss carryforward, perhaps limited in number of years,

-need to use idea of inventories that age.



Compressible Inventories, Limits on Inventory Change Rates

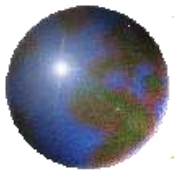
Natural gas distributors may store gas in compressed form in large reservoirs in preparation for cold weather and related demand spikes. There are two interesting features because of the pressure in the reservoirs, there are limits on the rate at which inventory can be increased or decreased, and further,

- a) It gets harder to insert more gas when inventory is high.
- b) It gets harder to remove more gas when inventory is low.

Key point:

On a cold day when the demand is high, we may be in trouble, even if we have a sufficient supply of gas, but the pressure is not high enough to withdraw it as rapidly as needed.

There are various approaches...



Compressible Inventories, Limits on Inventory Change Rates

Variables:

$INV(t)$ = inventory at end of period t ,

$ADD(t)$ = amount added in period t ,

$RMV(t)$ = amount removed in period t ,

A linear approximation:

Amount we can remove per period increases with amount remaining:

$$RMV(t) \leq a_R + f_R * INV(t-1);$$

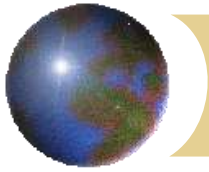
Amount we can add per period decreases with space remaining:

$$ADD(t) \leq a_A + f_A * (CAP - INV(t-1));$$

And of course:

$$INV(t) = INV(t-1) + ADD(t) - RMV(t),$$

$$INV(t) \leq CAP ;$$



Inventories with Gains or Losses

Gains examples: Investments, rabbits,...

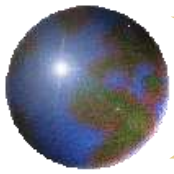
Loss examples: Work force attrition, medical radio-isotope decay,
water that evaporates, retail inventory breakage and shrinkage, ...
Spoilage rate for produce in supermarket may approach 0.4 fraction.

Parameter: $r (\geq -1)$ is the rate of gain per period, -1 means lose entire investment.

Balance equations:

$$INV(t) = (1+r)*INV(t-1) + PROD(t) - SELL(t);$$

Admonition: Do not declare $INV(t)$ to be an integer variable, e.g., if it represents number people in work force. You will probably get a “No feasible solution” message. You can, however, declare the $PROD(t)$ and $SELL(t)$ variables to be integer.



Inventories that Age, Perishable Products

Product perishes - cannot be sold after P periods.

Blood bank (21 days); food products: produce, milk, meat, cheese; cut flowers; pharmaceuticals; forest plots: pine, christmas trees; wine, whiskey;

Interesting decisions: When should we harvest/sell trees, cheese, ...?

Variables:

$INV(a, t)$ = inventory of age a at end of period t ,

$SELL(a, t)$ = amount sold/harvested/used of age a during period t ,

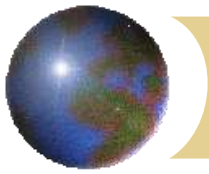
$PROD(t)$ = amount produced in period t ;

Balance equations:

$INV(1, t) = PROD(t) - SELL(a, t)$;

for $a = 2, 3, \dots, P$:

$INV(a, t) = INV(a-1, t-1) - SELL(a, t)$;



Inventories that Age: The When to Harvest Problem

Many products gain (or lose) value as they grow.

Big question: When should we harvest, taking into account:

Change in value vs. cost of keeping/growing the product for another week,
Demand for product of a given type or age.

Examples:

Cattle: Veal, hogs, when does gain/lb. of feed peak?

Current demand for each type?

Produce: Baby lettuce vs. full maturity lettuce.

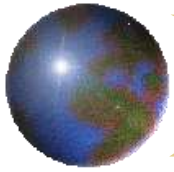
Cheese: young mild cheddar vs. old sharp cheddar.

Milk: Convert to cheese, yogurt, butter and store?

Trees: pine, Christmas trees.

When best to replace by young fast growing trees?

Wine, whiskey: Value increases with age, but worth the storage cost?



Precedence and Resource Constraints

There are precedence constraints among activities in a number of situations:

Mining-open pit/cast,

Block j cannot be removed until all blocks above removed.

Petroleum extraction,

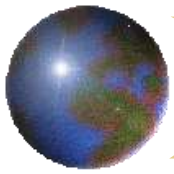
Stage j of extraction with associated production rate cannot be entered until stage $j - 1$ completed.

Air Traffic Congestion Modeling,

A plane cannot enter sector j of flight until it finished sector $j - 1$.

Tank scheduling in process industries,

A chemical batch cannot enter tank j of process until finishes $j - 1$.



Multi-Period Optimization Models, Resource Extraction

Notable feature: Cost/unit extracted and production rate from a location depends upon cumulative production from the location. Cost/unit may start high and then drop. In petroleum, cost/unit may start low, and then increase.

Mining: Partition ore area into blocks, and define

Parameters:

PP = set of predecessors pairs.

(i, j) in PP means j cannot be extracted before i is completed.

Decision variables:

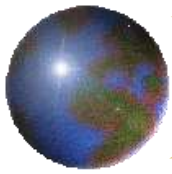
$z(j, t) = 1$ if block j is extracted in period t , else 0;

Crucial constraints:

For all (i, j) in PP :

Cannot extract block j unless i already extracted,

$$z(j, t) \leq \sum_{s \leq t} z(i, s);$$

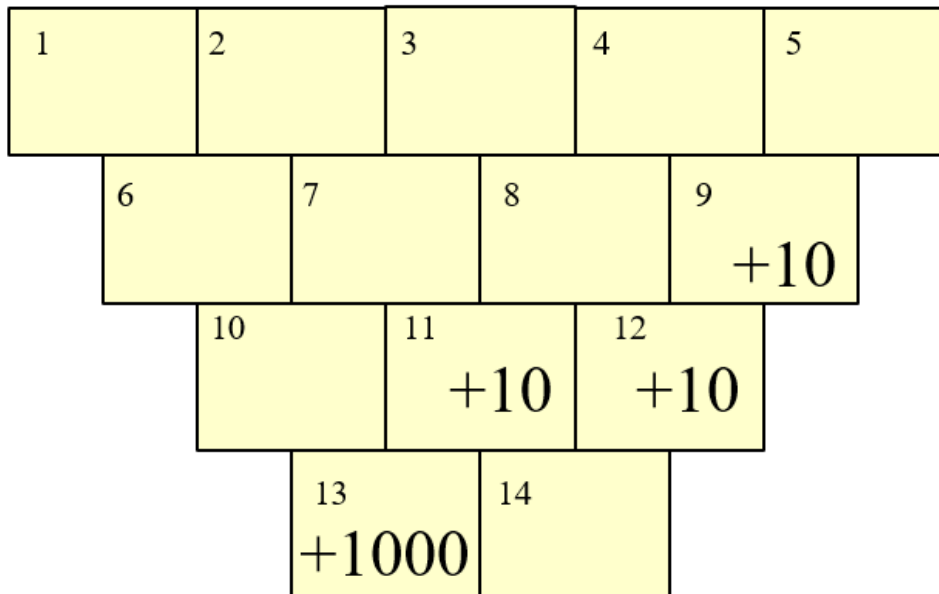


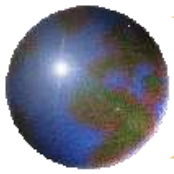
Multi-Period Optimization Models, Resource Extraction

Mining: Partition ore area into blocks.

In what sequence should we remove blocks if we can remove only two blocks/year.

Does it depend on your discount factor?





Precedence Constraints: How to Represent

Usual notation:

$z(j, p) = 1$ if task j is started in period p , else 0.

Precedence: If i must be started no later than j ,

$$\begin{aligned} z(j, p) + z(j, p-1) + \dots + z(j, 1) &\leq \\ z(i, p) + z(i, p-1) + \dots + z(i, 1), &\quad \text{for } p = 1, 2, \dots; \end{aligned}$$

Better: Define the cumulative “done” variable:

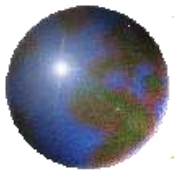
$$\begin{aligned} w(i, p) &= 1 \text{ if task } i \text{ is started in period } p \text{ or earlier, else 0.} \\ &= z(i, p) + z(i, p-1) + \dots + z(i, 1), \\ &= z(i, p) + w(i, p-1), \end{aligned} \quad \text{****}$$

Precedence: If i must be started no later than j ,

$$w(j, p) \leq w(i, p), \quad \text{for } p = 1, 2, \dots; \quad \text{****}$$

If there is a lead/processing time of λ :

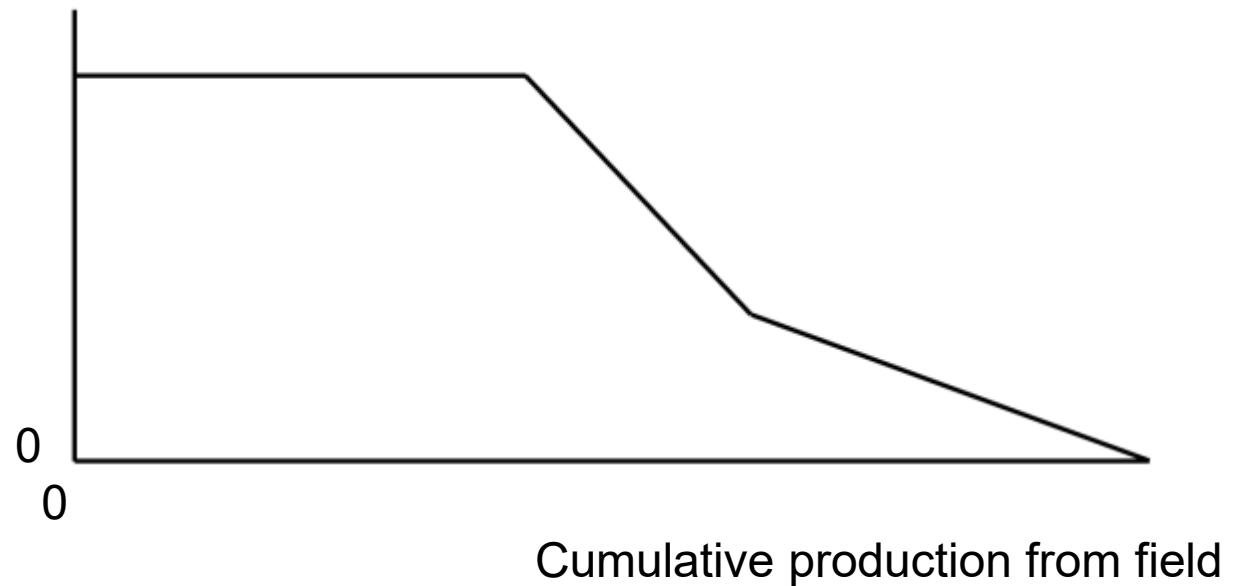
$$w(j, p) \leq w(i, p - \lambda),$$

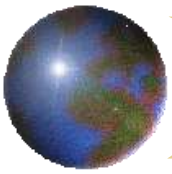


Multi-Period Optimization Models, Resource Extraction

Petroleum: Useful production rate goes down as field is depleted:

Maximum
production
rate from
a field





Multi-Period Optimization Models, Resource Extraction

Petroleum: Partition extraction from a field into a number of stages (e.g., 2 to 4).

Define Parameters:

$PU(i,s)$ = maximum production per period if field i is in stage s ,

$PT(i,s)$ = total cumulative production in field i available in stage s ,

Decision variables:

$P(i,t,s)$ = actual production in field i in period t in stage s ,

$z(i,t,s)$ = 1 if field i , in period t , enters stage s ; presume a field can enter at most one stage per period.

Constraints:

If we start stage s in period t , then previous stage must be exhausted:

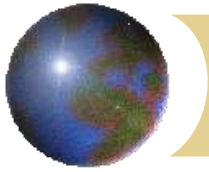
$$\sum_{r \leq t} P(i,r,s-1) \geq PT(i,s-1) * z(i,t,s);$$

Cannot produce more from a stage than available up to period t :

$$\sum_{r \leq t} P(i,r,s) \leq PT(i,s) * \sum_{r \leq t} z(i,r,s);$$

Cannot produce more than is available per period:

$$P(i,t,s) \leq PU(i,s) * \sum_{r \leq t} z(i,r,s);$$



Resource Constraints, Continuous vs. Discrete Time

For each job and machine combination, there is a processing time of the job on the specific machine, as well as a value of this assignment,

So:

Assign jobs to machines, and find a

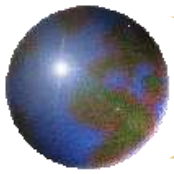
Sequence of jobs on each machine so that

At most one job is assigned to a specific machine at a specific instant,

and

Each job is done in its time window, and the

Value of the assignments is maximized;



Resource Constraints, Continuous vs. Discrete Time

Job to Machine Assignment and Sequencing

Jobs (trucks, ships, airplanes, patients, hotel guests...)

arrive over time at a facility (terminal, harbor, hospital, hotel,...).

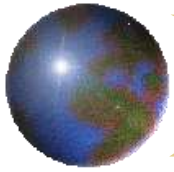
Facility has a number of “machines”: docks, gates, operating rooms....

Examples: airplanes to parallel runways (gates, de-icing stations) at an airport,
trucks or ships to docks at a freight terminal,
hotel guests to hotel rooms,
surgical procedures to operating rooms,
manufacturing jobs to machines in a factory.

Each machine can handle at most one job at a time.

A job cannot be started before its arrival time.

Each job has a due date by which its processing should be finished.



Resource Constraints, Continuous vs. Discrete Time

A One Machine Example

The “textbook” way of representing scarce resource/sequencing constraints

in continuous time is with pairwise sequencing constraints:

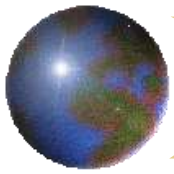
Sequence 20 tasks on one machine:

Continuous time formulation with “textbook” constraints: $\gg 60$ secs.

Discrete time formulation: < 1 sec.

Sequence 55 aircraft with time windows at 12 gates:

Discrete time formulation: 1 sec.



Resource Constraints, Continuous Time Formulation

```
! Sequence a set of tasks on a single machine (TaskSeqCont);
```

```
data:
```

```
! Each task has a processing time;
```

```
pt = 6 5 5 5 5    5 5 5 5 5  
    5 5 5 5 5    5 5 5 5 4;
```

```
BigM = 100;
```

```
enddata
```

```
! Continuous time formulation;
```

```
! Variables:
```

```
z(j,k) = 1 if task j precedes task k;
```

```
! Minimize the sum of completion times;
```

```
Min = @sum( task(t): ft(t));
```

```
! Minimize the maximum finish time;
```

```
! Min = ftmx;
```

```
@for( task( j):
```

```
ftmx >= ft( j);
```

```
ft( j) >= pt(j);
```

```
);
```

```
@for( txt( j,k) | k #gt# j:
```

```
! Either j precedes k ;
```

```
ft(k) >= ft(j) + pt(k)*z(j,k) - BigM* z(k,j);
```

```
! or k precedes j;
```

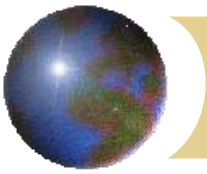
```
ft(j) >= ft(k) + pt(j)*z(k,j) - BigM* z(j,k);
```

```
! It must be one or the other;
```

```
z(j,k) + z(k,j) = 1;
```

```
@bin( z(j,k)); ! The z's are binary ( 0 or 1);
```

```
@bin( z(k,j)); );
```



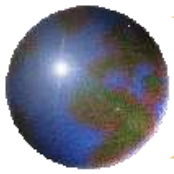
Resource Constraints, Discrete Time Formulation

```
! Sequence a set of tasks on a single machine (TaskSeqDisc);
! Discrete time formulation;
! Variables:
  Z(p,t) = 1 if task t starts in period p, i.e., starts at time p-1 ;

! The finish time of task t;
@for( task( t):
  ft( t) = pt(t) + @sum( period( p): (p-1)*z(p,t));

! Task t must start in some period p;
  @sum( pxt(p,t): z(p,t)) = 1;
);

! At most one task can be in process in period p.;
@for( period( p):
! Task t is in process in period p if it started in the interval [ p-pt(t)+1, p];
  @sum( pxt( p1, t) | p1 #ge# p-pt(t)+1 #and# p1 #le# p: z(p1,t)) <= 1;
);
```



Resource Constraints, Discrete Time Formulation

The notation for resource constraints is simplified if we introduce:

Parameters:

$pt(i)$ = processing time for task i , in number of periods,

$u(i, r)$ = amount of resource r required during each period that activity i is in process.

$cap(r)$ = amount of resource r available each period.

Variables:

$run(i, t)$ = 1 if activity i is in process during period t , else 0.

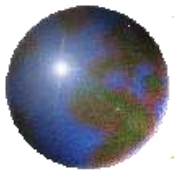
$z(i, t)$ = if activity i is started processing during period t ,

Then $run(i, t)$ is related to $z(i, t)$ by:

$$run(i, t) = run(i, t - 1) + z(i, t) - z(i, t - pt(i))$$

And the resource constraint for resource r in period t is:

$$\sum_i u(i, r) * run(i, t) \leq cap(r);$$



Choosing an Objective Function

For planning > a year, Net Present Value (NPV) is typical choice.

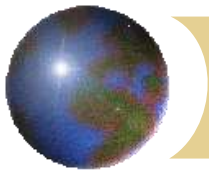
a) Simplest case: Choose a single interest rate r :

$$\text{Maximize } \sum_{t=0}^{t=N} c_t / (1 + r)^t$$

b) Period dependent rate. Estimate period dependent rates from prices of zero coupon bonds.

c) Risk adjusted period dependent rate.

d) Utility function if using scenarios.



Displaying Time Based Results:

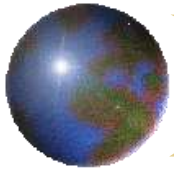
Space-Time Diagrams *

Gantt Charts *

Time Conversion/Calendar Routines *

**available in LINGO*

A classic treatment of how to presents results graphically, including Space-Time diagrams is Tufte (2001).



Displaying Time Based Results: Jet Taxi Routing Problem,

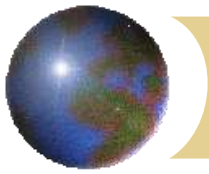
! The (Jet) Taxi Routing Problem.

Given a set of desired flights or trips to be covered,

figure out how to route planes/vehicles to cover these flights.

Repositioning/deadheading flights are allowed at a cost.

Sometimes called the Full-Truck-Load Routing problem.



Jet Taxi Routing Problem

! Scalar data;

VL, RP, RA, LA =

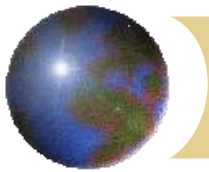
1 ! Relative value of covering a loaded flight;
0.05 ! Relative cost of a repositioning flight;
0.01 ! Relative cost of an aircraft;
2 ! Limit on total aircraft used;

! Vector data;

! The Cities, GMT offset, latitude, longitude, initial aircraft;

	City,	GMTOFF,	LATI,	LNGT,	INITA=	
! 1;	Chicago	-6	41.8500	-87.6500	0	! Chicago 6 hours behind Greenwich;
! 2;	Denver	-7	39.7392	-104.9903	1	! Denver is 7 hours ...;
! 3;	Tucson	-7	32.2217	-110.9258	1	
! 4;	Salt_Lake_City	-7	40.7500	-111.8833	1	
! 5;	Phoenix	-7	33.4833	-112.0667	0	
! 6;	Las_Vegas	-8	36.1667	-115.2000	0	
! 7;	Los_Angeles	-8	34.0522	-118.2428	1	

;



Jet Taxi Routing Problem

! The city pair trips available to cover/service;

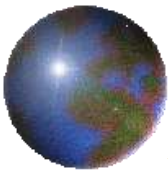
LODPAIR, Year, Month, Day, Hour, Minute =

!LEG	Origin City	Destination City	Year	Month	Day	Hour	Minute
1	Los_Angeles	Salt_Lake_City	2018	11	5	10	0
2	Salt_Lake_City	Phoenix	2018	11	6	14	20
3	Salt_Lake_City	Los_Angeles	2018	11	8	16	0
4	Phoenix	Chicago	2018	11	7	11	20
5	Salt_Lake_City	Las_Vegas	2018	11	9	16	0
6	Las_Vegas	Salt_Lake_City	2018	11	10	12	0
7	Tucson	Salt_Lake_City	2018	11	6	15	0
8	Denver	Las_Vegas	2018	11	7	8	30
9	Chicago	Phoenix	2018	11	8	10	30

! Get travel time matrix in minutes;

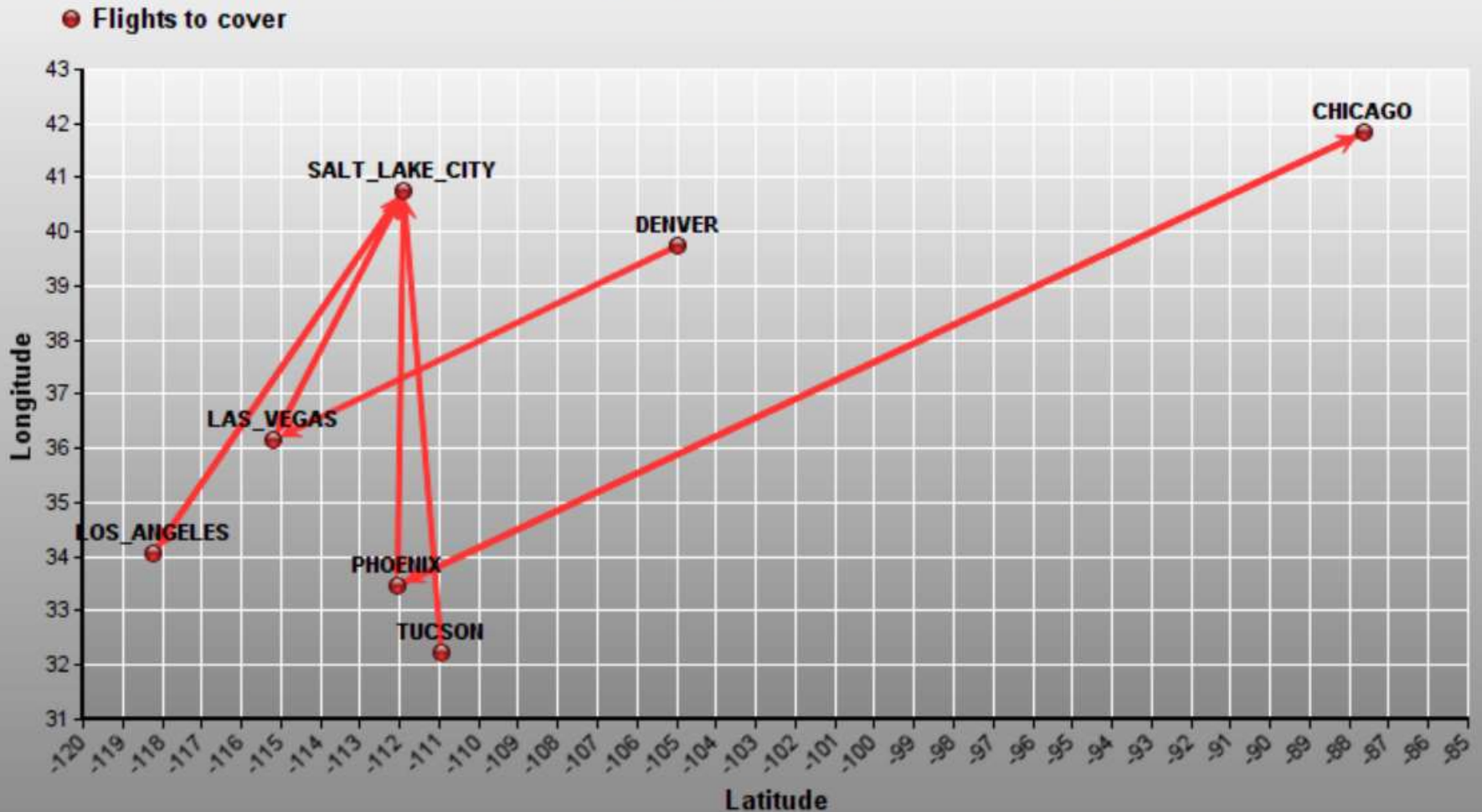
TRVTIM =

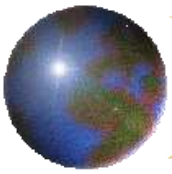
! Chi	Den	Tuc	SLC	Phn	LVg	LAX ;	
0	150	195	190	205	215	240	! Chicago;
150	0	115	85	120	115	155	! Denver;
195	115	0	120	60	95	120	! Tucson;
190	85	120	0	100	85	110	! Salt_Lake_City;
205	120	60	100	0	85	120	! Phoenix;
215	115	95	85	85	0	120	! Las_Vegas;
240	155	120	110	120	120	0	! Los_Angeles;;



Jet Taxi Routing Problem, Charting/Graphs

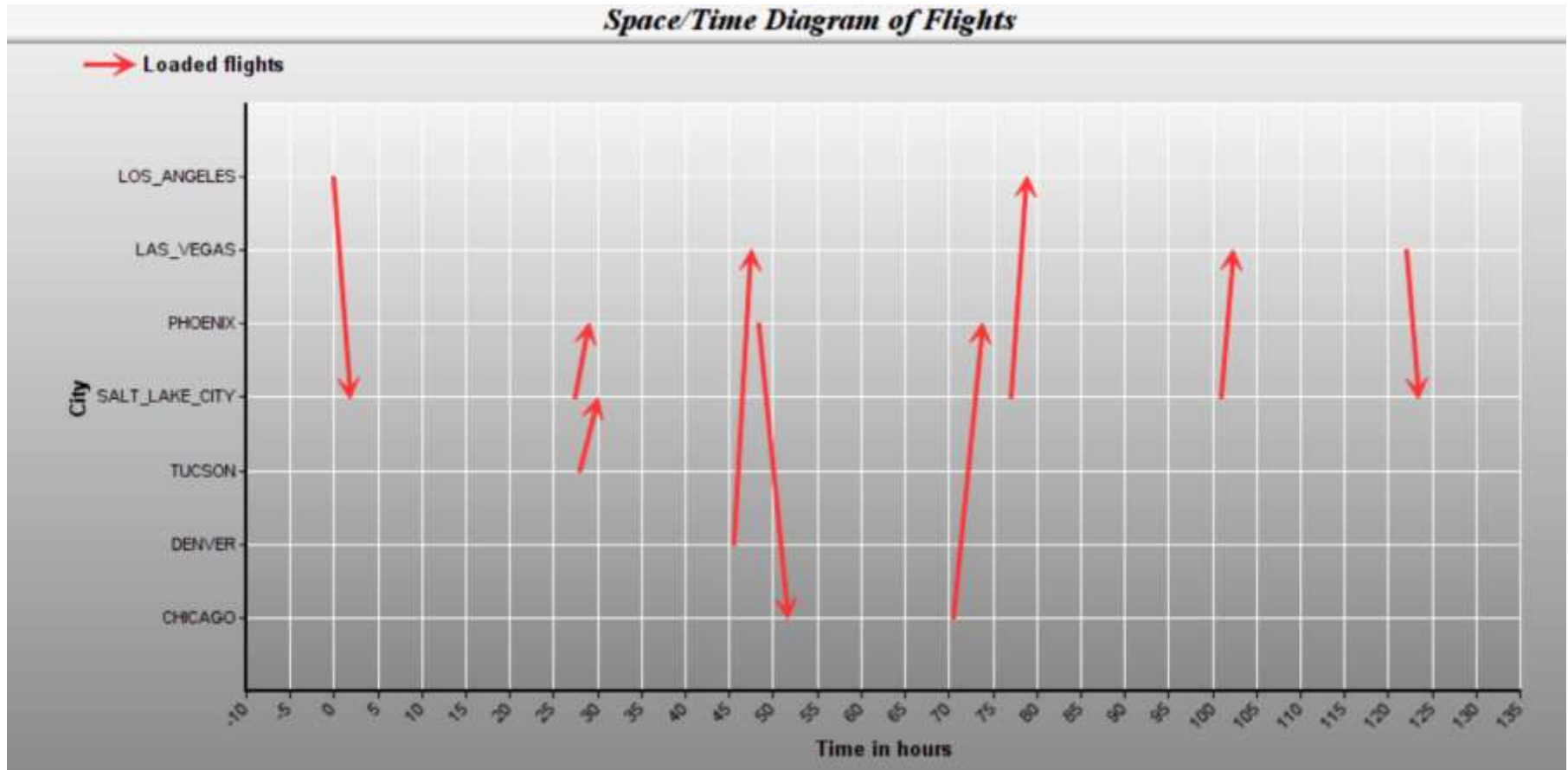
Latt/Long Display of Cities and Loaded Flights

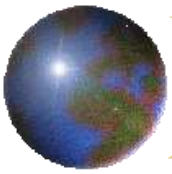




Jet Taxi Routing Problem

How many aircraft would you need to cover all these 9 flights?



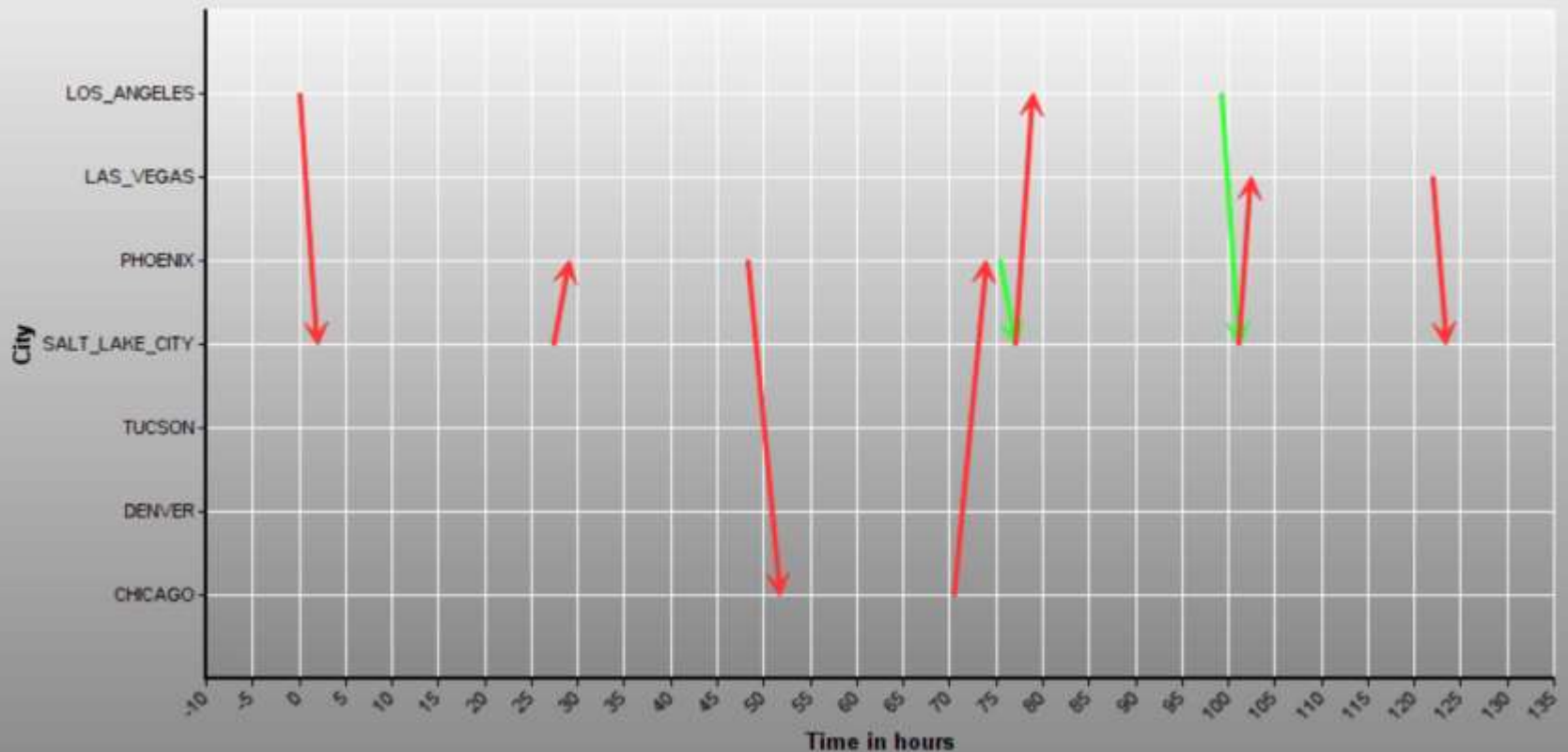


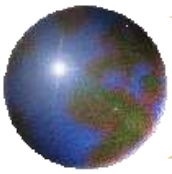
Jet Taxi Routing Problem

If you have only 1 aircraft, can cover 7 flights:

Space/Time Diagram of Flights

→ Loaded flights → Reposition flights

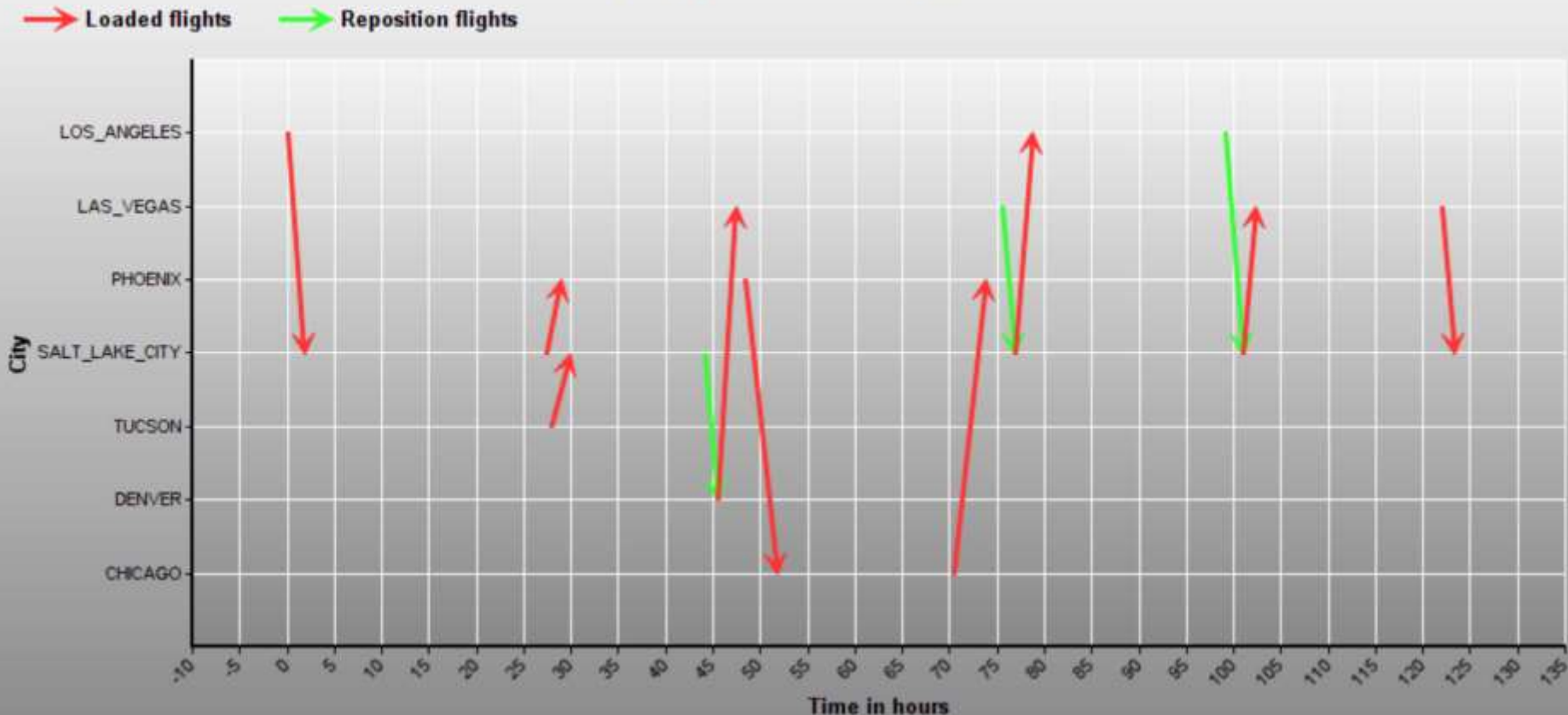


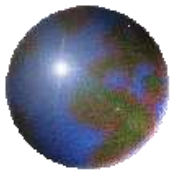


Jet Taxi Routing Problem

If you have 2 aircraft, can cover all 9 flights:

Space/Time Diagram of Flights





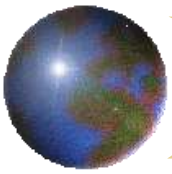
Jet Taxi Routing Problem

Loaded flights selected:

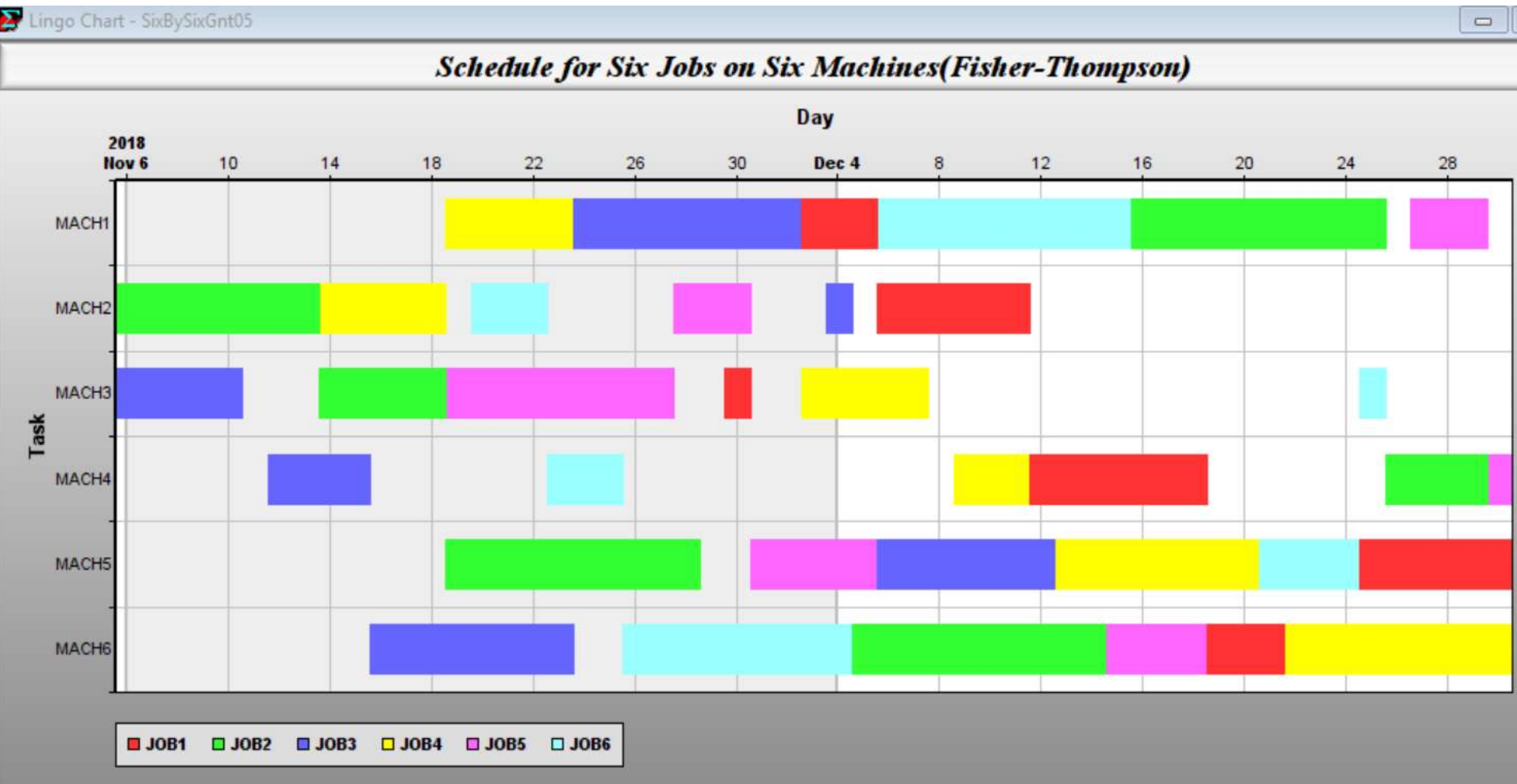
Origin	Destination	Depart at(local time)					
		yyyy	mm	dd	hr	mm	dwk
LOS_ANGELES	SALT_LAKE_CITY	2018	11	5	10	0	MON
SALT_LAKE_CITY	PHOENIX	2018	11	6	14	20	TUE
TUCSON	SALT_LAKE_CITY	2018	11	6	15	0	TUE
DENVER	LAS_VEGAS	2018	11	7	8	30	WED
PHOENIX	CHICAGO	2018	11	7	11	20	WED
CHICAGO	PHOENIX	2018	11	8	10	30	THU
SALT_LAKE_CITY	LOS_ANGELES	2018	11	8	16	0	THU
SALT_LAKE_CITY	LAS_VEGAS	2018	11	9	16	0	FRI
LAS_VEGAS	SALT_LAKE_CITY	2018	11	10	12	0	SAT

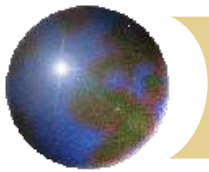
Repositioning Flights:

Origin	Destination	yyyy	mm	dd	hr	mm	dwk
PHOENIX	SALT_LAKE_CITY	2018	11	8	14	20	THU
LAS_VEGAS	SALT_LAKE_CITY	2018	11	8	13	35	THU
SALT_LAKE_CITY	DENVER	2018	11	7	7	5	WED



Gantt Charts are Helpful





It's About Time: How to Represent,- Display

Two Useful tools in LINGO for working with time:

a) **Calendar <=> Scalar time routines,**

! Convert calendar time to scalar time measured in seconds;

```
stimelv = @YMD2STM( yr, mo, da, hr, mn, sc);
```

! Convert scalar time to calendar time;

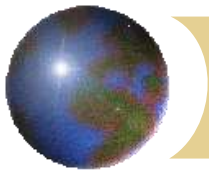
```
yr, mo, da, dwk, hr, mn, sc = @STM2YMDHMS( stimehn);
```

b) **Space-Time Diagrams.**

Show behavior over time.

Display time horizontally, locations vertically.

```
@CHARTSPACETIME( 'Space/Time Diagram of Flights',  
'Time in hours', 'City',  
'Loaded flights',  
  LODPAIRA, ! OD Pair list 1;  
  DLATIME,  ! Origin time list 1;  
  ALATIME,  ! Destination time list 1;  
'Reposition flights',  
  RPAIRU,  ! OD Pair list 2;  
  DUTIME,  ! Origin time list 2;  
  AUTIME); ! Destination time list 2;
```



Uncertainty, How to Represent?

Range of methods are used.

a) Stochastic Optimization/Programming is most comprehensive, “gold standard”:

Attach a probability to each possible scenario.

Optimize to find a policy that maximizes expected utility.

Challenge: Problem size grows exponentially with number of periods.

b) “Robust optimization” variations, though approximate, easier to use and to compute:

Identify a modest number of worrisome scenarios.

Add constraints so solution is repairable under worrisome scenarios.

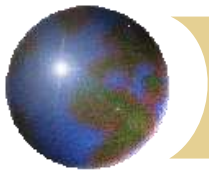
Examples of (a): Capacity planning at GM, some investment companies.

Examples of (b): “Spinning reserve” in electricity generation,

Spare capacity in resource extraction to meet surprise demand,

Reserve fuel on a flight to isolated airport,

avoiding crew changes in airline schedules (Yen and Birge).



Uncertainty, Kalman Filter

Kalman filter

(KalmanFilter.Ing)

“I skate to where the puck is going to be.”

-Wayne Gretzky

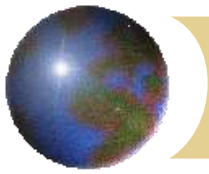
We have a system described at time t by a state vector: $x(t)$;

Examples:

1) GPS device: We want to predict vehicle location. When it loses GPS signal, we can update estimated location only by dead reckoning. When GPS signal is recovered, it gives noisy estimate of true state. The Combine step of Kalman filter combines the extrapolated and measured estimates.

2) Radar target estimation: Radar signal gives estimate of x & y distance. True state is (x, y) , (dx, dy) , & (ddx, ddy) . We get a fresh radar estimate of the location every second or so, depending on weather conditions.

How do we update estimated state of object and where it is going?



Uncertainty, Markov Decision Process (MDP)

Situation:

(RepairMDP.Ing)

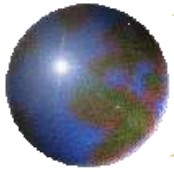
Period after period:

- A) "Nature" makes a random change to the system state,
- B) We spend some money to change the state in reaction to what "nature" did.

What is our optimal policy for (B)?

Examples:

	<u>System state</u>	<u>Random Effects</u>	<u>Decisions</u>
A)	Levels of various inventories.	Demands.	Restocking of various inventories.
B)	Condition of a section of road.	Deterioration due to traffic, weather.	Types of repairs.

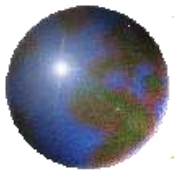


Uncertainty, MDP

Limitations:

State set and decision set must be discrete and finite.

Number of states grows multiplicatively with number of dimensions, e.g., number of different inventories.



Uncertainty, MDP: The Model

Parameters:

(RepairMDP.Ing)

$TP (s1, d, s) = \text{Conditional Prob(next state = } s \mid \text{ state is } s1, \text{ we make decision } d)$

$COST(s, d) = \text{cost of making decision } d \text{ if state is } s;$

Variables:

$Y(s, d) = \text{fraction of periods in state } s \text{ and make decision } d.$

Minimize average cost per period;

$$\text{MIN} = \sum_{s, d} COST(s, d) * Y(s, d);$$

Probabilities must sum to 1;

$$\sum_{s, d} Y(s, d) = 1;$$

For each state s :

{ Prob(state s) = Prob(all state $s1$ & decision d combinations);

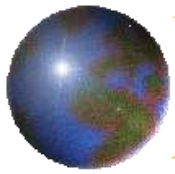
$$SSPROB(s) = \sum_{s, d} Y(s, d);$$

Prob{ being in state s } = sum of probabilities of getting there from other states $s1$;

$$SSPROB(s) =$$

$$\sum_{s1, d, s} Y(s1, d) * TP(s1, d, s);$$

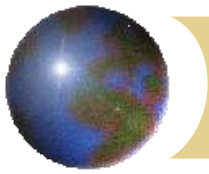
};



Uncertainty, Kalman Filter

Applications:

Trajectory estimation in GPS, radar, robotics/industrial control.



Uncertainty, Kalman Filter

How do we update estimated state of object and where it is going?

The state transition (and prediction) formula in matrix form is:

$$x(t) = F*x(t-1) + B*u(t) + e(t), \quad \text{where}$$

F = matrix that specifies how the state changes if no controls are applied,

$u(t)$ = vector of controls that have some effect on the state,

B = matrix that specifies how controls affect the state

$e(t)$ = random noise vector with covariance matrix Q ,

Q = covariance matrix that we re-estimate at each step.

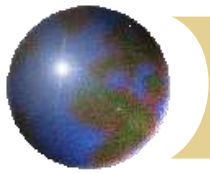
We may not have a perfect view of the state.

$z(t)$ = vector of what we measure/see. It is related to the true state by:

$$z(t) = H* x(t) + w(t), \quad \text{where}$$

H = matrix that relates how the true state is related to what we see,

$w(t)$ = random vector with covariance matrix R ;



Uncertainty, Choosing a Utility Function with Scenarios

If w = ending wealth (a random variable):

1) Maximize $E(w) - \alpha * SD(w)$. The Value@Risk risk measure is a variation of this, i.e.,
Maximize t , s.t., $\text{Prob}\{w \leq t\} \leq \gamma$. (Gives odd results if there are high cost/low probability outcomes.)

2) Downside risk: Maximize $\alpha * E(w) - E(\max(0, t-w)) + \gamma * t$.
where t is a tolerance or target for minimum acceptable wealth.
Conditional Value@Risk is a variation of this in which t is allowed to vary.

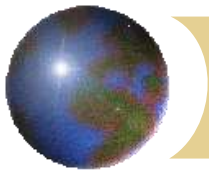
3) Power Law Utility Function:
Maximize $\text{Utility}(w) = E(w^\gamma)$; (Sometimes a scaling is applied: w^γ/γ):
It is strictly monotonic increasing (more is better) for $0 < \gamma$.
It is strictly concave (2nd \$ not as valuable as 1st \$) for $\gamma < 1$.

4) Log Utility Function: (Ed Thorp supposedly used this for bet sizing in Blackjack).
Maximize $E(\ln(w))$;

Log utility is a limiting case of power utility as γ decreases to 0.

To see this, note that the

marginal value of the power utility proportionally approaches $1/w$,
which is the same as marginal value with the log utility.



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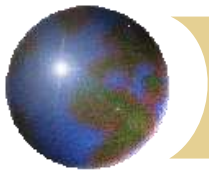
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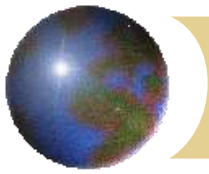
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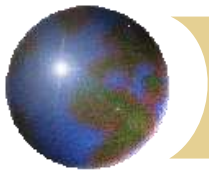
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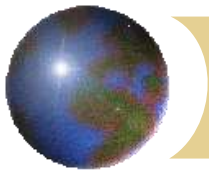
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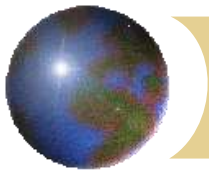
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Thank you for your attention!

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