

Optimization
Modeling with

LINGO

Fifth
Edition

LINDO SYSTEMS INC.



TRADEMARKS

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Fifth edition 2002
Printed in the United States of America
Second printing 2003

ISBN: 1-893355-00-4

Published by



LINDO SYSTEMS INC.

1415 North Dayton Street
Chicago, Illinois 60622
Technical Support: (312) 988-9421
<http://www.lindo.com>
e-mail: tech@lindo.com

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Preface

This book shows how to use the power of optimization, sometimes known as mathematical programming, to solve problems of business, industry, and government. The intended audience is students of business, managers, and engineers. The major technical skill required of the reader is to be comfortable with the idea of using a symbol to represent an unknown quantity.

This book is one of the most comprehensive expositions available on how to apply optimization models to important business and industrial problems. If you do not find your favorite business application explicitly listed in the table of contents, check the very comprehensive index at the back of the book.

There are essentially three kinds of chapters in the book:

1. introduction to modeling (chapters 1, 3, 4, and 19),
2. solving models with a computer (chapters 2, 5), and
3. application specific illustration of modeling with LINGO (chapters 6-18).

Readers completely new to optimization should read at least the first five chapters. Readers familiar with optimization, but unfamiliar with LINGO, should read at least chapters 2 and 5. Readers familiar with optimization and familiar with at least the concepts of a modeling language can probably skip to chapters 6-18. One can pick and choose from these chapters on applications. There is no strong sequential ordering among chapters 6-18, other than that the easier topics are in the earlier chapters. Among these application chapters, chapters 11 (on integer programming), and 12 (on stochastic programming) are worthy of special mention. They cover two computationally intensive techniques of fairly general applicability. As computers continue to grow more powerful, integer programming and stochastic programming will become even more valuable. Chapter 19 is a concluding chapter on implementing optimization models. It requires some familiarity with the details of models, as illustrated in the preceding chapters.

There is a natural progression of skills needed as technology develops. For optimization, it has been:

- 1) Ability to solve the models: 1950's
- 2) Ability to formulate optimization models: 1970's
- 3) Ability to use turnkey or template models: 1990's onward.

This book has no material on the mathematics of solving optimization models. For users who are discovering new applications, there is a substantial amount of material on the formulation of optimization models. For the modern "two minute" manager, there is a big collection of "off-the-shelf", ready-to-apply template models throughout the book.

Users familiar with the text *Optimization Modeling with LINDO* will notice much of the material in this current book is based on material in the LINDO book. The major differences are due to the two very important capabilities of LINGO: the ability to solve nonlinear models, and the availability of the set or vector notation for compactly representing large models.

Acknowledgments

This book has benefited from comments and corrections from Egon Balas, Robert Bosch, Angel G. Coca Balta, Sergio Chayet, Richard Darst, Daniel Davidson, Robert Dell, Hamilton Emmons, Saul Gass, Tom Knowles, Milt Gutterman, Changpyo David Hong, Kipp Martin, Syam Menon, Raul

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Negro, David Phillips, J. Pye, Fritz Raffensperger, Rick Rosenthal, James Schmidt, Paul Schweitzer, Shuichi Shinmura, Rob Stubbs, David Tulett, Richard Wendell, Mark Wiley, and Gene Woolsey and his students. The outstanding software expertise and sage advice of Kevin Cunningham was crucial. The production of this book (from editing and formatting to printing) was ably managed by Sarah Snider.