Approximations to Average Inventory in Supply Chain Design Models

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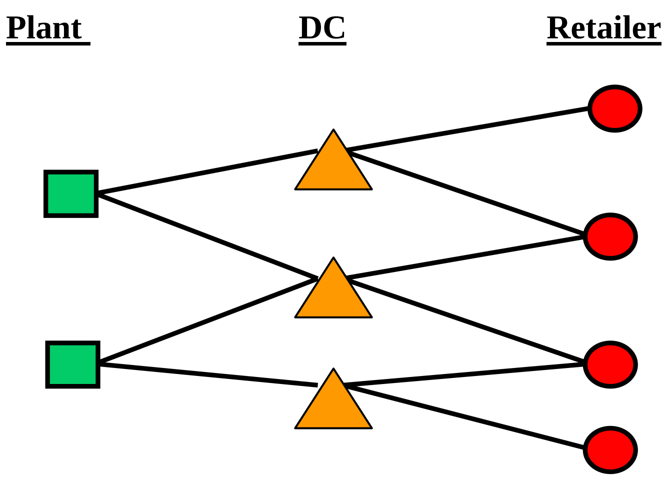
A popular supply chain design model is the single period, three level, Plant-to-Distribution center-to-Retailer model introduced by Geoffrion and Graves(1974) in their study of the Hunt-Wesson Foods distribution system. Early versions of these models were mainly concerned with tradeoffs between: a) increased fixed costs if you have more distribution centers(DC's) and b) increased outbound(DC to retailer) transportation costs if you have fewer DC's.

Figure 1. Generic 3-Level Distribution System



Little attention has been given to the effect of system design on the amount of inventory in the system. Here we try to model the effect of distribution system design on inventory in the system in a one period supply chain model. For small supply chains one can use a multiperiod model and explicitly track the flow of material through the system and compute inventory costs directly each period, see for example, the model used by Arntzen, Brown, Harrison, and Trafton(1995) for a computer manufacturer. In a one period model, however, one must estimate what the average inventory level is in each portion of the system. For an example of a comprehensive one period supply chain model, see the model LINDO_SCM_PlannerI.lng available in the Applications Library at http://www.lindo.com.

We consider the impact of system design on six types of inventory:

1) Pipeline inventory in transit from plant to distribution center(DC),
2) Cycle inventory at a DC, i.e., inventory resulting from the fact that deliveries to the DC tend to be in big lumps, like railcar lots, while withdrawals from the DC are in smaller lumps spread out over time.
3) Safety stock at a DC to protect against long lead time from the plant and random demand from retailers served by the DC,
4) Pipeline inventory in transit from DC to retailer,
5) Cycle inventory at a retailer,
6) Safety stock at a retailer to protect against long lead time from the DC and random demand seen by the retailer.

We will consider a very simple one product version of the 3-level model as follows.
Decision variables:

x_{ij} = tons per month shipped from plant i to DC j,
y_j = 1 if a DC is located at j, else 0,
w_{jk} = tons per month shipped from DC j to retailer k,

Parameters:

t_{ij} = transportation cost/unit from location i to location j. Variable costs at a facility may also be incorporated in this coefficient,
S_i = supply available at plant i per month,
F_j = fixed cost/month of having a DC at location j,
U_j = upper limit on throughput/month at DC location j,
D_k = demand at retailer k per month,
L_{ij} = lead time from facility i to facility j.

The model is:

Min $\Sigma_i\Sigma_j\, t_{ij}\, x_{ij} + \Sigma_j\, F_j\, y_j + \Sigma_j\Sigma_k\, t_{jk}\, w_{jk}$

Supply constraints at each plant i:
$\Sigma_j\, x_{ij} \le S_i$;

Flow balance at each DC j:
$\Sigma_i\, x_{ij} = \Sigma_k\, w_{jk}$;

Force DC k open:
$\Sigma_i\, x_{ij} \le U_j\, y_j$

Demand constraints at each customer k:
$\Sigma_j\, w_{jk} = D_k$;

For each DC k:
y_j = 0 or 1;

Common complications to this model are a) multiple products, b) multiple transportation modes, c) single sourcing constraints on retailers, and d) supply chain length constraints on the total lead time from plant through DC to retailer. The inventory approximations below are easy to generalize to these complications.

Plant-to-DC Pipeline Inventory

The total Plant-to-DC pipeline inventory is

$\Sigma_i \Sigma_j L_{ij} x_{ij}$;

If DC's are scattered uniformly over the market region, then we might expect the total pipeline inventory from plants to DC's to be relatively independent of the system design. See Geoffrion(1976) for some of the arguments. Thus, as a first approximation, we might be justified in disregarding plant-to-DC pipeline inventory. Given its simple form, however, it is not very painful to include it, and it would steer solutions towards ones that have DC's uniformly distributed over the market region.

Cycle Inventory at a DC

As the volume through a DC increases, we might expect that the average cycle inventory should increase less than proportionately with the volume. If the economic order quantity(EOQ) model is applicable, then the cycle inventory at a DC should be proportional to the square root of the volume through the DC. For example, if the volume through a DC is increased by a factor of 4, the EOQ model says we will meet this volume by ordering twice as much per order, and ordering twice as frequently. Thus, the average cycle inventory doubles.

Standard supply chain planning models like to use linear approximations to various costs. That is, we would like to approximate cycle inventory a DC j by a function of the form $a_j y_j + b_j \Sigma_i x_{ij}$. Suppose we know that the cycle inventory at DC j is about 25,000 at throughput volume of 10,000 units, and we would like the approximation to also be accurate at a throughput of 20,000 units. According to the EOQ model, the cycle inventory cost at 20,000 units should be about 1.414*25,000 = $35,355. Thus, we want to solve:

$a_j + b_j 10000 = 25{,}000$, and

$a_j + b_j 20000 = 35{,}355$.

Solving, we get that $b_j = 1.0355$, and $a_j = 14645$.

Thus, the amount of cycle inventory at DC j would be modeled by

$14645\, y_j + 1.0355\, \Sigma_i x_{ij}$.

Safety Stock at a DC

Simple models of safety stock say that the amount of safety stock in a DC should be proportional to the standard deviation in the demand faced by the DC and to the square root of the lead time. If demands are independent, then the standard deviation in demand faced by the DC should be proportional to the square root of the volume. If there is a single supplier for the DC, then we might expect that the safety stock should be proportional to $(L_{ij}\, x_{ij})^{0.5}$.

Suppose that for plant i, DC j, the lead time is 0.15 months and at a volume of 10,000 units the safety stock inventory is 4000 units. We would also like the approximation to be accurate at both a volume of 10,000 units and at 20,000 units. Thus, we want to solve:

$\alpha_j + \beta_j*(.15)^{0.5} *10000 = 4{,}000$, and

$\alpha_j + \beta_j*(.15)^{0.5} *20000 = 2^{0.5} *4{,}000 = 5{,}657$.

Solving, we get that $\beta_j = .4278$, and $\alpha_j = 2343.1$. Thus, safety stock at DC j would be modeled by:

$2343.1*y_j + .4278*\Sigma_i\, L_{ij}^{0.5}\, x_{ij}$.

Notice that a supplier i with a longer lead time to j will cause j to carry more safety stock.

<u>DC-to-Retailer Pipeline Inventory</u>

We expect DC-to-Retailer pipeline inventory to increase as the number of DC's is decreased because outbound lead times will tend to increase. The obvious expression for total DC-to-Retailer pipeline inventory is:

$\Sigma_j\Sigma_k\, L_{jk}\, w_{jk}$;

<u>Cycle Inventory at a Retailer</u>

The argument is similar to that for DC's. It differs in that in a typical study the demand or volume at a retailer is fixed and known in advance. Which DC supplies the retailer may not be known in advance, and to the extent that the lead time effects the order size, the choice of supplier may effect the average cycle stock. If we again use the EOQ model for guidance, the average cycle stock is proportional to the square root of the fixed cost of placing an order. Suppose the cost of a delivery to a retailer is a fixed cost per trip proportional to the distance but not a function of the volume delivered on the trip. Then you could argue that the fixed cost of a delivery is proportional to the lead time. If DC j supplies retailer k, then the cycle stock at k should be proportional to the square root of L_{jk}. As an example, suppose that the lead time from DC j to retailer k is 0.1 months, and retailer k currently has $2000 in cycle inventory. Thus, we want to solve:

$\gamma_k *(.1^{0.5}) = 2{,}000$

The solution is $\gamma_k = 2000/.31623 = 6325$.

For example, suppose the monthly volume at retailer k is 3000 units, then the amount of cycle inventory at retailer k would be modeled by:

$(6325/3000)\ \Sigma_j\ (L_{jk}^{0.5})*w_{jk}$.

Safety Stock at a Retailer

If demands from day to day at a retailer are independent then the amount of safety stock the retailer should carry to achieve a specifed level of service should be approximately proportional to the square root of the lead time that the retailer faces. Considering our same retailer as before, suppose with a lead time of 0.1 months he carries a safety stock of $1200. Thus, we want to solve:

$$\delta_k *(.1^{0.5}) = 1200.$$

Solving, we get

$$\delta_k = 1200/.31623 = 3795.$$

For example, suppose the monthly volume at retailer k is 3000 units, then the amount of safety stock at retailer k would be modeled by:

$$(3795/3000)\ \Sigma_j\ (L_{jk}^{0.5})w_{jk}.$$

All the above are estimates of the average level or value of inventory. If one is minimizing the average cost per month, then the appropriate cost of capital should be applied to these expressions when including them in the objective.

Approximation Quality

Where a linear approximation is used above to approximate a nonlinear cost function, one should check that the solution obtained is in fact in the region where the linear approximation is expected to be accurate. Even if the solution is in the region where the approximation is accurate, it is not a guarantee that the solution is in fact close to optimal for the original nonlinear problem. Alternatively, one can improve on the approximation quality by using a piecewise linear approximation to the square root of a volume rather than a simple fixed plus linear. This is easy to do in the original formulation by introducing a second DC choice for location j. So for example, DC $j1$ might have a cost curve that is accurate over the volume range [5000, 10000], whereas DC $j2$ might have a cost curve that is accurate over the volume range [10000, 15000]. We would have to add a mutual exclusivity constraint, $y_{j1} + y_{j2} \leq 1$. The capacity of DC $j1$ would be set to U_{j1} = 10000.

Yet another alternative is to leave the nonlinear square root terms in the model explicitly. Global optimization software has gotten to the point that it can now reliably solve nontrivial size models to guaranteed global optimality.

Supply Chain Length and Inventory Constraints

A slightly different way of avoiding the bad effects of long lead times is to prohibit long lead times, long supply lines, or large inventories. For example, if you wish to constrain the average lead time into DC j to be no greater than 0.5, you can do this with the constraint

$$\Sigma_i L_{ij} x_{ij} \leq 0.5 \Sigma_i x_{ij}$$

For high technology products, there may be a risk of product obsolescence and so one may wish to have a limit on total inventory in the system. If you wish to put a limit I_{max} on total inventory in the system, you can have a constraint of the form:

$$\begin{aligned} & \Sigma_i\Sigma_j L_{ij} x_{ij} \\ & + \Sigma_j a_j y_j + \Sigma_j b_j \Sigma_i x_{ij} \\ & + \Sigma_j \alpha_j * y_j + \Sigma_j \beta_j * \Sigma_i L_{ij}^{0.5} x_{ij} \\ & + \Sigma_j \Sigma_k L_{jk} w_{jk} \\ & + \Sigma_k (\gamma_k / D_k) \Sigma_j (L_{jk}^{0.5}) w_{jk} \\ & + \Sigma_k (\delta_k / D_k) \Sigma_j (L_{jk}^{0.5}) w_{jk} \leq I_{max} ; \end{aligned}$$

If you wish to prohibit a long supply line from a plant, through a DC, and on to a customer, you can use the "path formulation" originally proposed by Geoffrion and Graves(1974). Define the variables:

z_{ijk} = tons per month shipped from plant i through DC j and on to customer k.

If the total lead time along the path ijk is greater than some acceptable lead time, then that variable z_{ijk} is set to zero and removed from the problem. In the original problem formulation, replace every occurrence of x_{ij} by $\Sigma_k z_{ijk}$ and every occurrence of w_{jk} by $\Sigma_i z_{ijk}$. This may result in a dramatic increase in the number of variables.

References

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