Solving Multi-object Auctions with LP/IP

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1 November 2001

Abstract

In many auctions/markets, the value of an object to a participant depends significantly on what other objects the participant does or does not acquire. This dependence is of two main types: complements and substitutes. For the complements case in particular, there has been some concern that such auctions are too difficult to solve exactly. Some of these problems have even been (inaccurately) described as intractable. We describe the use of linear programming(LP/IP) based methods for solving in reasonable time, such multi-object auctions as encountered in practice.

1. Introduction to Multi-object/Combinatorial/Combinational Auctions

In many auctions/markets, the value of an object to a participant depends significantly on what other objects the participant acquires. This dependence is of two main types: complements and substitutes. Objects A and B are complements if the value of A to a buyer is greater if the buyer also acquires B. They are substitutes if the value to the buyer of A is decreased if the buyer also acquires B.

When there are such interactions among objects being auctioned, the traditional one-at-a-time style of auction may be very inefficient. By inefficient, we mean that the allocation of objects to bidders by the auction may be dominated in some sense by some other allocation. As an example, consider a simple real estate auction in which two parcels, A and B are to be sold. For some reason, e.g., to build a large factory, Bidder X needs both parcels and is willing to pay $200,000 for the pair. Suppose parcel A is sold first. Bidder X might drop out when the price for A reaches $110,000 because he guesses that Parcel B will sell for a similar amount and thus the pair will be beyond X's budget. In fact, Parcel B sells for $20,000. There are several possible endings to this little example: i) bidder X may have left after the sale of A, never the wiser of the missed opportunity, or ii) after the auction, bidder X buys Parcel A from its buyer for, say, $130,000 and Parcel B from its buyer for, say, $60,000. Clearly, X and the two other original buyers are happier with this outcome. The original seller may have missed $70,000 in revenue because of a poorly designed auction. The point of this little example is to motivate the observation that "solving" a multi-object auction is a nontrivial problem.

There is substantial literature on multi-object auctions. One of the first to succinctly lay out some of the issues are Rassenti, Smith, and Bulfin(1982). A more recent published analysis of what makes multi-object auctions difficult appears in Rothkopf, Pekec, and Harstad(1998)(where the term "combinational auction" is suggested). An even more recent excellent survey is the working paper by de Vries and Vohra(2000). Additional relevant pieces of published research will be listed later in pertinent sections.

We analyze the use of linear programming(LP/IP) based methods for solving such multi-object auctions. We specifically cover: a) examples of multi-object auctions in practice, b) the problem of solving for an optimal allocation of objects, and c) the pricing problem of deciding how much to charge the winners and how to explain to the losing bidders why they lost. There has been concern that multi-object auctions cannot be solved exactly in practice. The FCC for example in its sale of radio spectrum bandwidth has been very reluctant to use the multi-object format. We show empirically that for a wide range of practical multi-object auctions, the problem is computationally manageable.

2. Applications of Multi-object Auctions

Multi-object auctions have sometimes been considered "intractable". In fact a number of them have been solved regularly since 1980. Some examples are:

Crude Oil Sales
Sales of six different types of crude oil at the Elks Hills Petroleum reserve, see Jackson and Brown(1980). This seems to be the first published use of LP/IP to solve a multi-object auction.

Course registration
At University of Chicago, solved as MIP/LP since 1981. See Graves, Sankaran, and Schrage(1993). This seems to be the first use of large scale LP to solve an auction.

Vacation Time Slots in an Organization
Members of an organization bid on vacation timeslots in combinations of 2 one-week slots/person. See Strevell and Chong(1985)

Multi-period Pipeline Capacity
Pipeline capacity is sold in capacity intervals, e.g., ten days. The allocation is solved with MIP. See Knowles(1996). An interesting feature of this application was the handling of alternate optima for the allocation of objects.

Real Estate Combinatorial Auctions
Typical forms of bids are a) "1 of n" bids for substitutes, and b) "all of n" bids for complements. There is little published here, but computers have been used almost since the availability of portable computers.

Airport runway timeslots
A take-off slot has no value without a matching landing slot elsewhere. It is not clear that LP/IP has ever been used for this, however, it is the setting that motivated the original paper by Rassenti, Smith, and Bulfin(1982).

Job Interview Time Slot Bidding
Some business schools, e.g., the University of Chicago, “sell” job interview slots to the highest bidding students. Each student receives a fixed amount of points with which to bid on interview slots for the season. The combinatorial aspect arises because a student bidding on several companies can be awarded a set of time slots only if they are non-overlapping in time.

Radio Frequency Spectrum
The FCC sold rights to radio frequencies in various regions of the country. Because of the complementarity of nearby licenses it was solved as a set of simultaneous single object auctions but with “participation” constraints to help avoid some of the difficulties.

Bundle Pricing by a Seller
A pricing problem dating at least to Stigler(1963) is how much discount should a seller give if a buyer buys two or more products, e.g., wordprocessor + spreadsheet software. Hanson and Martin(1990) describe the use of LP/IP to several a general version of this problem.

Vendor Selection
Manufacturers who ship lots of product by truck have the problem of deciding which vendors to use, e.g., trucking carriers, given the discounts they offer for giving them more business. Moore, E., J. Warmke, and L. Gorban(1991) describe the use of a MIP at Reynolds Metal to decide which trucking carrier bids to accept for a large set of specified shipments. See also Camm, Knepper, and Offhaus(2000).

Market Clearing with Fixed Costs
A number of e-commerce sites are trying to tackle the problem: given multiple sellers, buyers, products, and transportation carriers, and their associated bids, who buys what from whom and shipped by what carrier?

3. Solving an Auction

It is useful to think of an auction of consisting of the sequence:

1) Allocation Problem: Who gets what? A major consideration is that we want to get at least a Pareto optimum, i.e., there is no other allocation in which no one is worse off, and at least someone is better off.

2) Pricing problem: How much do winners pay? How do we explain to losers why they lost? How do we set prices so that bidders can and will bid efficiently?

3.1 Solving the Allocation Problem

A number of methods have been suggested for deciding which bidder gets which object, e.g.,

a) Sequential Single Auctions, e.g., first sell A,
then B, then C, etc.
Traditional. May produce very non-Pareto Optimal allocations.

b) Simultaneous Single Auctions.
Exciting, because there are several things going on at once.
Early bidders have exposure risk.
FCC Radio spectrum used this with "continuous participation" constraints.

c) Two stage.
1) single auctions, as in (a) above, followed by
2) all one bundle auction. It may still be non-Pareto optimal.

d) Simultaneous, all bundles.
Perplexing, but can be good. Used in some real estate auctions.

e) Sealed simultaneous, all bundles.
Solve as an IP.

<u>Example of Two-stage Auction:</u>
Two stage auctions have been used in real estate sales. Suppose two parcels, A and B are to be sold. Suppose there are five potential bidders with preferences as follows:

Example:

Bidder:	U	V	W	X	Y	
Reservation Price(RP):	120	70	60	30	20	≤ 1
Object A)	1	1		1		≤ 1
Object B)	1		1		1	≤ 1

E.g. Bidder U is willing to pay up to 120 for both objects A and B. Bidder Y wants only object B and is willing to pay at most 20 for it.

In the actual auction on which this example is based was a sequential auction. First object A was (tentatively) sold; then object B was (tentatively) sold, finally, bids were accepted on the bundle A and B.

A plausible outcome would be that object A would be sold for 31, then B for 21, and finally A and B together would be sold to U for 71(to beat out V). One could argue that the seller should have gotten at least 121, almost twice as much, by selling A to V and B to W.

If run as a simultaneous auction, suppose U bids 100. V and W might individually be unwilling to make up the difference. Each would hope the other would make up the difference.

3.2 A Generic Auction IP

Define:

Parameters:

b_{ij} = amount bid by bidder i for his jth bundle,

R_k = units available of resource k,

a_{ijk} = units required of resource k by bidder i's jth bid,

Decision variables:

x_{ij} =1 if bid ij is successful, else 0.

$$Max \sum_i \sum_j b_{ij} x_{ij}$$

s.t.

For each resource k:

$$\sum_i \sum_j a_{ijk} x_{ij} \leq R_k$$

For each bidder i:

$$\sum_j x_{ij} \leq 1$$

3.3 Computational Difficulty of Auction Problems

Rothkopf, Pekec, and Harstad(1998) use complexity analysis to identify a class of multi-object auctions that are guaranteed to be easily solveable. Their main result(among others) is: If the bidding model is a network LP, then it can be solved in polynomial time.

Loosely speaking, sale objects that are substitutes lead to network problems(which are easy), whereas complements lead to set-packing-like problems(which in theory can be hard).

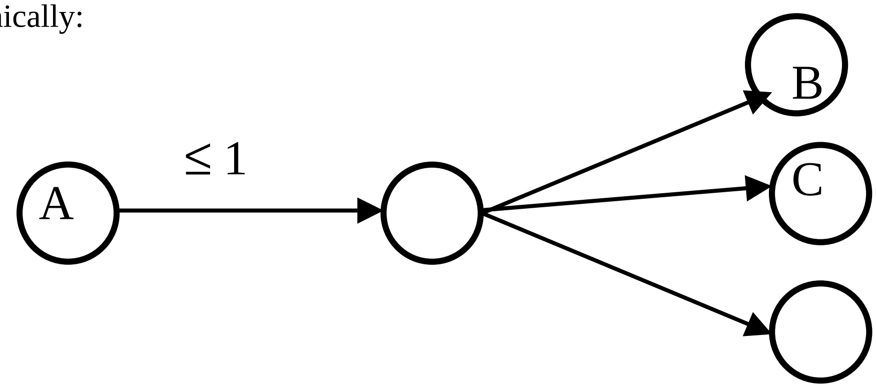
Empirically, single seller, multi-object auctions seem to be solvable in reasonable time. Market clearing auctions with multiple buyers and sellers, + fixed transaction costs take a long time to solve the allocation problem to a proven optimal solution, however, a 1% solution is obtainable in a few seconds.

3.4 Bids that Result in Network LP's

Some real estate auctions allow "1 out of *n*" bids, i.e., "Here are *n* bids, award me at most one". This leads to a network LP. Since 1981, The University of Chicago GSB registration system allows bundle bids, "1 out of n" and swap bids, e.g., "I will sell object A if I can get exactly one of objects B, C, for prices….."

The swap is a generalization of a "1 of *n*" bid.

Graphically:



"1 of n" bids help reduce exposure risk.

Exposure Risk in Multi-Object Auctions(When Run as Single Object Auctions)

There are a number of reasons why single object auctions are unsatisfactory for multi-object settings.

One difficulty is exposure risk. A bidder may be able to complete only part of his desired multi-object transaction. In the following example, bidder X wants to buy <u>both</u> A and B.

Example:

	Bidder			
	X	Y	Z	
	9	7	7	
Object A)	1	1		≤ 1
Object B)	1		1	≤ 1

If object A sells first, bidder X will acquire it for $8, but then be outbid on B. Thus, bidder X is left owning a useless A, which he might try to resell to Y, who might have meantime bought something else.

The FCC auction, run as a simultaneous single object auction, tried the make it more difficult for Z above to "ambush" X, by introducing a "participation" constraint to force Z to reveal his intentions immediately.

Problems with Participation Constraint when there are Substitutes

The participation constraint(used in FCC auctions) says: You can bid on object A in round *t+1* only if either:

i) you made a "real" bid on A in round *t*, or
ii) you had the winning bid on A at start of round *t*.

A "real" bid is defined as one that is higher than the winning bid in the previous round. The above mechanism provides some protection against exposure risk in a simultaneous single object auction for a bidder who wants two or more objects. The participation constraint does not work well if there are substitutes. Consider the sale of two objects. Bidder X is willing to pay up 8 for either of the objects A or B. He wants only one. Suppose the reservation prices(not publicly known) are:

	X1	X2	Y1	Z1	
Reservation price:	8	8	9	5	
Object A)	1		1		≤ 1
Object B)		1		1	≤ 1
Just one)	1	1			≤ 1

When the prices of each object reaches 4, bidder X has to make a decision to drop out of the bidding for one of the objects. If he drops out of object B, it will sell for 5 to Z. Bidder Y will get object A for 9. Thus, the seller will sell object B for 5, even though bidder X is willing to pay 8 for it.

4. Some Computational Results for Auctions

Below we summarize computational experience with a number of actual multi-object auctions.

Example: Course registration auction, Spring term of 2000

350 object types(course sections),
2091 bidders,
84176 bids, 3.8 objects/bid,
48 seconds to solve.(Using LINDO callable library, 300 Mhz)

Solution comparison with heuristic solution:

	IP	Heuristic	%
Total value awarded	25,507,457	23,555,500	8.3
Successful bidders	2062	1949	5.8

Example: Same bid set, different capacities

390 object types(course sections),
18 seconds to solve.

Exampe: Real Estate, artificial data

50 objects, one each,
50 bidders,
200 bids, 3.23 objects/bid,
38 seconds to solve.

Example: Real Estate, artificial data

20 objects, one each,
60 bidders,
240 bids, 3.15 objects/bid,
4 seconds to solve.

Example: Interview Placement Bidding

The formulation is:

Maximize value of interviews scheduled;
subject to:
For each company * timeslot combination:
Number of interviews scheduled ≤ interviewers available;
For each job candidate * company combination:
Number interviews scheduled ≤ 1;
For each job candidate and any two overlapping time slots:
Number interviews scheduled ≤ 1;

Some typical results are:

265 bidders(students desiring interview slots),
129 companies interviewing during the week,
1056 objects available(interview slots),
936 objects requested(interviews desired by students)

12,321 rows in the integer program,
10,051 columns in the integer program,
817 objects granted.
820.5 = interviews granted in continuous LP relaxation.
28 seconds to solve to optimality on a 600 MHz computer.

Example: Interview Placement Bidding

448 bidders(students desiring interview slots),
135 companies interviewing during the week,
1751 objects available(interview slots),
42,403 rows in the integer program,
41,384 columns in the integer program,
1660 objects requested(interviews desired by students)
1580.166 = interviews granted in continuous LP relaxation.
1579 objects granted.
111 seconds to solve to optimality on a 600 MHz computer.

Example: Selling Pipeline Capacity

See Knowles(1996)

Data and variables:

Parameters:

R_k = units of capacity available on day k,
b_j = amount bid per daily unit of capacity in bid j,
s_j = start date of bid j,
t_j = end date of bid j,
a_{kj} = 1 if bid j wants capacity on day k, else 0
(implied by s_j and t_j),
l_j = lower limit on amount accepted if >0,
u_j = upper limit on amount accepted/day,

Decision variables:

x_j = units of daily capacity allocated to bid j.
y_j = 1 if bid j is accepted, else 0.

Formulation:

$$Max \sum_j b_j (t_j + 1 - s_j) x_j$$

s.t.

For each day k:

$$\sum_j a_{kj} x_j \le R_k ,$$

$$l_j\, y_j \;\le\; x_j \le\; u_j\, y_j,$$
$$y_j = 0 \text{ or } 1.$$

An interesting feature of the specific application described by Knowles(1996) was that if there were multiple optimal solutions, one must choose the average solution, to the extent possible.

A typical problem had number of bids < 100, with a solution time < minute.

Example: Trucking Core Carrier Problem

The use of bidding to select trucking carriers was described by Moore, Warmke, and Gorban(1991) at Reynolds Metals, and for firms in the furniture and chemical industries, by Camm et.al.(2000).

Formulation:

Parameters:

D_{ij} = number of loads to be moved on from i to j; "lane ij"

b_{cij} bid of carrier c on lane ij.

Decision variables;

X_{cij} = number of loads by carrier c from i to j;

Y_{ci} = 1 if carrier c serves origin i;

Z_c = 1 if c is a core carrier;

Objective

Min $\Sigma_{cij}\, b_{cij}\, X_{cij}$;

Each lane i,j must be covered:

$$\Sigma_c\; X_{cij} = D_{ij};$$

For each origin i, turn on carriers serving it:

$$X_{cij} \;\le\; D_{ij}\; Y_{ci}$$

Turn on carrier c if used at any origin:

$$Y_{ci} \;\le\; Z_c;$$

Limit on number of core carriers:

$$car_{min} \le\; \Sigma_c\, Z_c \;\le car_{max};$$

Limit on number of carriers at origin i:

$$carmin_i \le\; \Sigma_c\, Y_{ci} \le carmax_i\,;$$

Allow Non-core Carriers to Carry at Most Fraction *p* of Loads

Define:
W_{cij} = loads carried by non core carrier *c* on lane *i*,*j*;

Modified demand constraint:
Each lane *i*,*j* must be covered:

$$\Sigma_c X_{cij} + \Sigma_c W_{cij} = D_{ij} ;$$

Loads carried by non-core carriers:

$$\Sigma_{cij} W_{cij} \leq p\Sigma_{ij} D_{ij};$$

Solution Times:

We have solved a number of variations of this problem for a logistics firm. For example it takes about ten seconds to solve a problem with 17 carriers, 356 lanes, 2694 bids, and number of core carriers limited to a dozen. For unrelated larger problems a typical time is 500 seconds for 27 to 49 carriers, 703 to 1473 lanes, resulting in 32,549 to 40,431 variables, and 33,205 to 41,879 constraints.

Example: Market Clearing Model

Define:

D_{jk} = amount needed by customer *j* of commodity *k*;

B_{jk} = bid by customer *j* per unit of commodity *k*;

S_{ik} = supply available from vendor *i* of commodity *k*;

A_{ik} = ask price of vendor *i* per unit of commodity *k*;

U_{ij} = shipping capacity from vendor *i* to customer *j*;

C_{ij} = ask price per unit from *i* to *j*;

Decision variables:

x_{ijk} = amount shipped from *i* to *j* of commodity *k*.

Market Clearing Model, Formulation:

Maximize Producer/Consumer/Carrier Surplus:

$$Max \sum_i \sum_j \sum_k (B_{jk} - A_{ik} - C_{ij}) x_{ijk}$$

For each supplier *i* and commodity *k*,

ships no more than is available:

$$\sum_j x_{ijk} \le S_{ik}$$

For each buyer *j* and commodity *k*,

receives no more than requested:

$$\sum_i x_{ijk} \le D_{jk}$$

For each supplier/buyer lane *ij*,

Carries no more than its capacity:

$$\sum_k x_{ijk} \le U_{ij}$$

This is an LP and thus easy to solve.

Market Clearing Model, Extensions:

Some IP extensions to the market clearing model:

- Fixed cost for using
  - A specific lane,
  - A specific lane & commodity(picking charge)
- Single sourcing
  - Of all commodities,
  - For each commodity.

For a single customer,

the above extensions seem not too difficult.

For multiple suppliers and customers, it is not too difficult to get a 1% solution, however, it can be challenging to get a guaranteed optimum.

E.g., a 20x20 fixed charge transportation problem

can be difficult to solve to optimality.

These statistics from the above examples support the argument that real multi-object auctions can be properly solved in reasonable time.

4. Solving the Pricing Problem

Once the allocation problem has been solved, we must decide a) how much to charge the winners, and b) how to explain to losers why they lost. For single object auctions both of these are fairly straight forward. A standard consideration is that it would be nice if the prices are as incentive compatible as possible, i.e., bidders are motivated to bid their true value for a bundle. There are two benefits of this: a) we are more justified in solving for the maximum benefit allocation, and b) bidders need waste a lot of research on finding out what other bidders will bid, they simply bid their so-called “reservation prices”.

An obvious question for a multi-object auction is: Is there an incentive compatible pricing mechanism, that is, a charging mechanism such that bidders are motivated to bid their true reservation prices? The answer is “Yes, although you may be reluctant to use it.” Vickrey(1961)’s “highest unsuccessful bid” mechanism works for single object type auctions. The general idea is that a winning bidder pays the amount by which his receiving his bundle of objects reduced the utility of all other bidders. This general charging or incentive scheme probably has been attributed to Clarke(1971) and Groves(1973).

More specifically our problem, suppose S is the set of all bidders, and we solve the original problem:

$$z(S) = Max \sum_i \sum_j b_{ij} x_{ij}$$

s.t.

For each resource k:

$$\sum_i \sum_j a_{ijk} x_{ij} \le R_k ,$$

For each bidder i:

$$\sum_j x_{ij} \leq 1,$$

$$x_{ij} = 0 \text{ or } 1.$$

Suppose bidder w is a winner with his bid b_{wj*}. We then remove bidder w from the problem, e.g., by replacing the 1 by a 0 in the last constraint for $i = w$ and resolve the problem to get $z(S\text{-}w)$. Also, define $z^{ij}(S)$ = solution to the original problem but with b_{ij} = 0 and the additional constraint that $x_{ij} = 1$. Thus, $z^{ij}(S)$ is the value received by bidders other than i when i is awarded his jth bundle. With bidder w winning, the other bidders received a value of $z^{wj*}(S)$. Thus, bidder w's injury to other bidders(and the price he pays) is $z(S\text{-}w)\text{-}\ z^{wj*}(S)$. Notice the crucial feature that the amount paid by w is independent of bidder w's own bid. The amount of the bid determines only whether or not he wins.

To see that this charging scheme motivates w to bid his true reservation prices, suppose w "shaves" his bid down. If he still gets combination $j*$, he still pays the same amount. So underbidding does not save w any money. On the otherhand, underbidding may cause him to not win, when in fact he could have gotten the bundle for his reservation price or less. Now suppose w bids higher than his reservation price, perhaps in the hope of winning when he would otherwise not, but still paying no more than his reservation price. The only time bidding higher than the true reservation price makes a difference is when the reservation price is not high enough to be successful. This means that if w's bid j were forced into the solution it would cost other participants more than b_{ij}. Thus, the only cases where bidding more than the reservation price makes a difference, is when the bidder would have to pay more than his reservation price. Summarizing, if you underbid, the only difference it can make is that sometimes you will not win, when you could have in fact got the desired bundle for your reservation price or less. If you overbid, the only diffence it can make is that sometimes you will win when you otherwise have not, but you will pay more than your reservation price. Bikhchandani, et. al.(2001) give a nice general, mathematical version of this argument.

Some reasons why one might not wish to use this scheme in practice are a) it may be computationally expensive. It looks like a possibly difficult integer program needs to be solved for at least each successful bidder, and b) it place prices on combinations of objects rather than individual objects. Some participants might find this non-traditional method of stating prices as non-intuitive, and possibly unacceptable.

Putting Prices on Individual Objects

Except for when quantity discounts are given, bidders are accustomed to per individual items rather than for arbitrary bundles. Let us consider the individual price problem.

One statement of it is find prices p_k and μ_i , and violations u_{ij}, v_{ij} :

$$Min\ \alpha \sum_R u_{ij} + (1-\alpha) \sum_A v_{ij}$$

For each rejected bid, *ij* in *R*:

$$\mu_i + \sum_k p_k a_{ijk} \geq b_{ij} - u_{ij}$$

For each accepted bid, *ij* in *A*:

$$\mu_i + \sum_k p_k a_{ijk} \leq b_{ij} + v_{ij}$$

See Rassenti et. al.(1982).

Incentive Compatibility

A sealed bid auction is incentive compatible if a participant is motivated to bid his true value(R.P. reservation price) for a bundle.

Result 1: A bidder is motivated to bid his R.P. if the amount of his bid determines whether he wins or not, but it does not determine how much he pays.

Result 2: An auction cannot be incentive compatible for all participants.

Example: Consider a (prospective) buyer and seller with respective reservation prices of 60 and 40.
How can you determine a clearing price, $40 \leq p \leq 60$ that is independent of the bid and ask prices?

The auction can be made incentive compatible for buyer by saying that a sale will occur if the buyer's offer is ≥ the seller's ask, and
the clearing price will be the seller's price.

Vickrey Prices

In a single object type Vickrey auction, the clearing price is the highest unsuccessful bid(in a highest bid wins auction).

Result: Vickrey prices are incentive compatible for buyers in single object auctions.

Result: Open auction price = Vickrey price + δ

Result: Dual prices from an appropriately perturbed LP are Vickrey prices.

A simpler attempted generalization of Vickrey prices to multi-object auctions is to set the $u_{ij} = 0$ and the constraints involving the accepted bids have strictly positive slack. (This may not always be possible.)

The Free-rider/Threshold Problem

There may be several unavoidable pricing problems in multi-object auctions. One is the game theory problem of the free rider/threshold problem.

<u>Example</u>

Bidder:	W	X	Y		Plausible Prices			
Reservation Price(RP):	70	60	40		<u>i</u>	<u>ii</u>	<u>iii</u>	<u>iv</u>
Item A)	1	1		≤ 1	60	30	35	42
Item B)	1		1	≤ 1	10	40	35	28

Clearly,
bidder X should get A, and
bidder Y should get B.

They should pay together, at least 70.

How much should each pay?

Bidder Y would prefer to have X pay 60,
Then Y pays 10 and gets a free ride.

There is an incentive for X and Y to not reveal their true reservation price.

In this regard, (35, 35) seems best.

The Subsidy Problem

The subsidy problem arises if one wishes to use individual object prices.

<u>Example</u>

Bidder:	W	X	Y		Plausible Prices
Reservation Price(RP):	70	60	50		
Object A)	1		1	≤1	50
Object B)	1	1		≤1	60
Object C)		1	1	≤1	0

Clearly, bidder W should get A and B. X and Y get nothing.

What should be the prices? How much should W pay?
70? 110? (Generalized Vickrey says 60).

If we insist upon prices on individual objects, then it seems the best answer is that the prices should be (50, 60) and W pays 70 to get a bundle with an "official" value of 110.

Seller Bundling:

All of the above examples involved buyers specifying preferences for bundles. There lots of examples, however, where the seller specifies the bundles. By selling a bundle of products for a discount, the seller may be able to increase revenue.

Example: Microsoft Office.

	ReservationPrice by Individual Product	
Bidder/Market segment:	X	Y
Spreadsheet)	$800	$400
Wordprocessor)	$400	$800

If the vendor sets a single market price for each product individually, the most he can make is $1600.

If, however, the vendor sells the two products only as a bundle for $1200, he can make $2400.

How should the seller do the bundling? It is an interesting optimization problem.

Formulation of Seller Bundling Problem

R_{ij} = reservation price of customer i for bundle j,
N_i = "size of customer" i (i.e., number of individual customers in segment i),
s_i = consumer surplus achieved by customer i,

Decision variables:

y_{ij} = 1 if customer i buys bundle j, 0 otherwise,
P_j = price of bundle j set by the seller.

We will treat the empty bundle as just another bundle, so we can say every customer buys exactly one bundle. "Stackelberg" equilibrium problem is:

$$\text{Maximize} \quad \sum_i \sum_j N_i \, y_{ij} \, P_j$$

subject to:

Each customer *i* buys one bundle:

$$\sum_j y_{ij} = 1;$$

For each customer *i*, its achieved consumer surplus is:

$$s_i = \sum_j (R_{ij} - P_j)y_{ij}$$

Customer *i* will buy only that bundle *j* for which its consumer surplus, s_i, is maximum. This is enforced by the constraints:

For each customer *i* and bundle *j*:

$$s_i \geq R_{ij} - P_j$$

$$y_{ij} = 0 \text{ or } 1$$

The solution time for a problem with 8 bundles(3 products), 4 customer segments, fixed market development costs, and discounts is less than a second.

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