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Game Theory and Cost Allocation

16.1 Introduction

In most decision-making situations, our profits (and losses) are determined not only by our decisions, but by the decisions taken by outside forces (e.g., our competitors, the weather, etc.). A useful classification is whether the outside force is indifferent or mischievous. We, for example, classify the weather as indifferent because its decision is indifferent to our actions, in spite of how we might feel during a rainstorm after washing the car and forgetting the umbrella. A competitor, however, generally takes into account the likelihood of our taking various decisions and as a result tends to make decisions that are mischievous relative to our welfare. In this chapter, we analyze situations involving a mischievous outside force. The standard terminology applied to the problem to be considered is *game theory*. Situations in which these problems might arise are in the choice of a marketing or price strategy, international affairs, military combat, and many negotiation situations. For example, the probability a competitor executes an oil embargo against us probably depends upon whether we have elected a strategy of building up a strategic petroleum reserve. Frequently, the essential part of the problem is deciding how two or more cooperating parties "split the pie". That is, allocate costs or profits of a joint project. For a thorough introduction to game theory, see Fudenberg and Tirole (1993).

16.2 Two-Person Games

In so-called two-person game theory, the key feature is *each of the two players must make a crucial decision ignorant of the other player's decision*. Only after both players have committed to their respective decisions does each player learn of the other player's decision and each player receives a payoff that depends solely on the two decisions. Two-person game theory is further classified according to whether the payoffs are constant sum or variable sum. In a constant sum game, the total payoff summed over both players is constant. Usually this constant is assumed to be zero, so one player's gain is exactly the other player's loss. The following example illustrates a constant sum game.

A game is to be played between two players called *Blue* and *Gold*. It is a single simultaneous move game. Each player must make her single move in ignorance of the other player's move. Both moves are then revealed and then one player pays the other an amount specified by the payoff table below:

Payoff from Blue to Gold

		Blue's Move	
		a	**b**
	a	4	–6
Gold's Move	**b**	–5	8
	c	3	–4

Blue must choose one of two moves, (*a*) or (*b*), while Gold has a choice among three moves, (*a*), (*b*), or (*c*). For example, if Gold chooses move (*b*) and Blue chooses move (*a*), then Gold pays Blue 5 million dollars. If Gold chooses (*c*) and Blue chooses (*a*), then Blue pays Gold 3 million dollars.

16.2.1 The Minimax Strategy

This game does not have an obvious strategy for either player. If Gold is tempted to make move (*b*) in the hopes of winning the 8 million dollar prize, then Blue will be equally tempted to make move (*a*), so as to win 5 million from Gold. For this example, it is clear each player will want to consider a random strategy. Any player who follows a pure strategy of always making the same move is easily beaten. Therefore, define:

BM_i = probability Blue makes move i, $i = a$ or b,
GM_i = probability Gold makes move i, $i = a$, b, or c.

How should Blue choose the probabilities BM_i? Blue might observe that:

If Gold chooses move (*a*), my expected loss is:
$4\ BMA - 6\ BMB$.

If Gold chooses move (*b*), my expected loss is:
$-5\ BMA + 8\ BMB$.

If Gold chooses move (c), my expected loss is:
$3\ BMA - 4\ BMB$.

So, there are three possible expected losses depending upon which decision is made by Gold. If Blue is conservative, a reasonable criterion is to choose the BM_i, so as to minimize the maximum expected loss. This philosophy is called the *minimax strategy*. Stated another way, Blue wants to choose the probabilities BM_i, so, no matter what Gold does, Blue's maximum expected loss is minimized. If *LB* is the maximum expected loss to Blue, the problem can be stated as the LP:

```
MIN = LB;
! Probabilities must sum to 1;
              BMA      + BMB  = 1;
! Expected loss if Gold chooses (a);
     -LB + 4 * BMA  - 6 * BMB <= 0;
! Expected loss if Gold chooses (b);
     -LB - 5 * BMA  + 8 * BMB <= 0;
! Expected loss if Gold chooses (c);
     -LB + 3 * BMA  - 4 * BMB <= 0;
```

The solution is:

```
Optimal solution found at step:        2
Objective value:               0.2000000
Variable           Value        Reduced Cost
      LB       0.2000000           0.0000000
     BMA       0.6000000           0.0000000
     BMB       0.4000000           0.0000000
     Row    Slack or Surplus      Dual Price
       1       0.2000000            1.000000
       2       0.0000000          -0.2000000
       3       0.2000000           0.0000000
       4       0.0000000           0.3500000
       5       0.0000000           0.6500000
```

The interpretation is, if Blue chooses move (*a*) with probability 0.6 and move (*b*) with probability 0.4, then Blue's expected loss is never greater than 0.2, regardless of Gold's move.

If Gold follows a similar argument, but phrases the argument in terms of maximizing the minimum expected profit, *PG*, instead of minimizing maximum loss, then Gold's problem is:

```
MAX = PG;
! Probabilities sum to 1;
          GMA     + GMB     + GMC  = 1;
! Expected profit if Blue chooses (a);
-PG + 4 * GMA - 5 * GMB + 3 * GMC >= 0;
! Expected profit if Blue chooses (b);
-PG - 6 * GMA + 8 * GMB - 4 * GMC >= 0;
```

The solution to Gold's problem is:

```
Optimal solution found at step:        1
Objective value:               0.2000000
Variable           Value        Reduced Cost
      PG       0.2000000           0.0000000
     GMA       0.0000000           0.1999999
     GMB       0.3500000           0.0000000
     GMC       0.6500000           0.0000000
     Row    Slack or Surplus      Dual Price
       1       0.2000000            1.000000
       2       0.0000000           0.2000000
       3       0.0000000          -0.6000000
       4       0.0000000          -0.4000000
```

The interpretation is, if Gold chooses move (*b*) with probability 0.35, move (*c*) with probability 0.65 and never move (*a*), then Gold's expected profit is never less than 0.2. Notice Gold's lowest expected profit equals Blue's highest expected loss. From Blue's point of view the expected transfer to Gold is at least 0.2. The only possible expected transfer is then 0.2. This means if both players follow the random strategies just derived, then on every play of the game there is an expected transfer of 0.2 units from Blue to Gold. The game is biased in Gold's favor at the rate of 0.2 million dollars per play. The strategy of randomly choosing among alternatives to keep the opponent guessing, is sometimes also known as a mixed strategy.

If you look closely at the solutions to Blue's LP and to Gold's LP, you will note a surprising similarity. The dual prices of Blue's LP equal the probabilities in Gold's LP and the negatives of

Gold's dual prices equal the probabilities of Blue's LP. Looking more closely, you can note each LP is really the dual of the other one. This is always true for a two-person game of the type just considered and mathematicians have long been excited by this fact.

16.3 Two-Person Non-Constant Sum Games

There are many situations where the welfare, utility, or profit of one person depends not only on his decisions, but also on the decisions of others. A two-person game is a special case of the above in which:

1. there are exactly two players/decision makers,
2. each must make one decision,
3. in ignorance of the other's decision, and
4. the loss incurred by each is a function of both decisions.

A two-person constant sum game (frequently more narrowly called a zero sum game) is the special case of the above where:

(4a) the losses to both are in the same commodity (e.g., dollars) and
(4b) the total loss is a constant independent of players' decisions.

Thus, in a constant sum game the sole effect of the decisions is to determine how a "constant sized pie" is allocated. Ordinary linear programming can be used to solve two-person constant sum games.

When (1), (2) and (3) apply, but (4b) does not, then we have a two-person non-constant sum game. Ordinary linear programming cannot be used to solve these games. However, closely related algorithms, known as linear complementarity algorithms, are commonly applied. Sometimes a two-person non-constant sum game is also called a bimatrix game.

As an example, consider two firms, each of which is about to introduce an improved version of an already popular consumer product. The versions are very similar, so one firm's profit is very much affected by its own advertising decision as well as the decision of its competitor. The major decision for each firm is presumed to be simply the level of advertising. Suppose the losses (in millions of dollars) as a function of decision are given by Figure 16.1. The example illustrates that each player need not have exactly the same kinds of alternatives.

Figure 16.1 Two Person, Non-constant Sum Game

		Firm A		
		No Advertise	Advertise Medium	Advertise High
FIrm B	No Advertise	-4 / -4	-3 / -2	-5 / 1
	Advertise	-1 / -5	-2 / -1	-1 / 0

Negative losses correspond to profits.

16.3.1 Prisoner's Dilemma

This cost matrix has the so-called prisoner's dilemma cost structure. This name arises from a setting in which two accomplices in crime find themselves in separate jail cells. If neither prisoner cooperates with the authorities (thus the two cooperate), both will receive a medium punishment. If one of them provides evidence against the other, the other will get severe punishment while the one who provides evidence will get light punishment, *if the other does not provide evidence against the first.* If each provides evidence against the other, they both receive severe punishment. Clearly, the best thing for the two as a group is for the two to cooperate with each other. However, individually there is a strong temptation to defect.

The prisoner's dilemma is common in practice, especially in advertising. The only way of getting to Mackinac Island in northern Michigan is via ferry from Mackinaw City. Three different companies, Sheplers, the Arnold Line, and the Star Line operate such ferries. As you approach Mackinaw City by car, you may notice up to a mile before the ferry landing, that each company has one or more small roadside stands offering to sell ferry tickets for their line. Frequent users of the ferry service proceed directly to the well-marked dock area and buy a ticket after parking the car and just before boarding the ferry (no cars are allowed on Mackinac Island). No reserved seats are sold, so there is no advantage to buying the tickets in advance at the stands. First time visitors, however, are tempted to buy a ticket at a company specific stand because the signs suggest that this is the safe thing to do. The "socially" most efficient arrangement would be to have no advanced ticket booths. If a company does not have a stand, however, while its competitors do, then this company will lose a significant fraction of the first time visitor market.

The same situation exists with the two firms in our little numerical example. For example, if *A* does not advertise, but *B* does, then *A* makes 1 million and *B* makes 5 million of profit. Total profit would be maximized if neither advertised. However, if either knew the other would not advertise, then the one who thought he had such clairvoyance would have a temptation to advertise.

Later, it will be useful to have a loss table with all entries strictly positive. The relative attractiveness of an alternative is not affected if the same constant is added to all entries. Figure 16.2 was obtained by adding +6 to every entry in Figure 16.1:

Figure 16.2 Two Person, Non-constant Sum Game

		Firm A		
		No Advertise	Advertise Medium	Advertise High
FIrm B	No Advertise	2 / 2	4 / 3	7 / 1
	Advertise	1 / 5	5 / 4	6 / 5

We will henceforth work with the data in Figure 16.2.

16.3.2 Choosing a Strategy

Our example illustrates that we might wish our own choice to be:

i. somewhat unpredictable by our competitor, and
ii. robust in the sense that, regardless of how unpredictable our competitor is, our expected profit is high.

Thus, we are lead (again) to the idea of a random or mixed strategy. By making our decision random (e.g., by flipping a coin) we tend to satisfy (i). By biasing the coin appropriately, we tend to satisfy (ii).

For our example, define a_1, a_2, a_3 as the probability A chooses the alternative "No advertise", "Advertise Medium", and "Advertise High", respectively. Similarly, b_1 and b_2 are the probabilities that B applies to alternatives "No Advertise" and "Advertise", respectively. How should firm A choose a_1, a_2, and a_3? How should firm B choose b_1 and b_2?

For a bimatrix game, it is difficult to define a solution that is simultaneously optimum for both. We can, however, define an equilibrium stable set of strategies. A stable solution has the feature that, given B's choice for b_1 and b_2, A is not motivated to change his probabilities a_1, a_2, and a_3. Likewise, given a_1, a_2, and a_3, B is not motivated to change b_1 and b_2. Such a solution, where no player is motivated to unilaterally change his or her strategy, is sometimes also known as a Nash equilibrium. There may be bimatrix games with several stable solutions.

What can we say beforehand about a strategy of *A*'s that is stable? Some of the a_i's may be zero while for others we may have $a_i > 0$. An important observation which is not immediately obvious is the following: the expected loss to *A* of choosing alternative *i* is the same over all *i* for which $a_i > 0$. If this were not true, then *A* could reduce his overall expected loss by increasing the probability associated with the lower loss alternative. Denote the expected loss to *A* by v_A. Also, the fact that $a_i = 0$ must imply the expected loss from choosing *i* is $> v_A$. These observations imply that, with regard to *A*'s behavior, we must have:

$2b_1 + 5b_2 \geq v_A$ (with equality if $a_1 > 0$),
$3b_1 + 4b_2 \geq v_A$ (with equality if $a_2 > 0$),
$b_1 + 5b_2 \geq v_A$ (with equality if $a_3 > 0$).

Symmetric arguments for *B* imply:

$2a_1 + 4a_2 + 7a_3 \geq v_B$ (with equality if $b_1 > 0$),
$a_1 + 5a_2 + 6a_3 \geq v_B$ (with equality if $b_2 > 0$).

We also have the nonnegativity constraints:

$a_i \geq 0$ and $b_i \geq 0$, for all alternatives *i*.

Because the a_i and b_i are probabilities, we wish to add the constraints $a_1 + a_2 + a_3 = 1$ and $b_1 + b_2 = 1$.

If we explicitly add slack (or surplus if you wish) variables, we can write:

$2b_1 + 5b_2 - slka1 = v_A$
$3b_1 + 4b_2 - slka2 = v_A$
$b_1 + 5b_2 - slka3 = v_A$
$2a_1 + 4a_2 + 7a_3 - slkb1 = v_B$
$a_1 + 5a_2 + 6a_3 - slkb2 = v_B$
$a_1 + a_2 + a_3 = 1$
$b_1 + b_2 = 1$
$a_i \geq 0$, $b_i \geq 0$, $slka_i \geq 0$, and $slkb_i \geq 0$, for all alternatives *i*.
$slka1 * a1 = 0$
$slka2 * a2 = 0$
$slka3 * a3 = 0$
$slkb1 * b1 = 0$
$slkb2 * b2 = 0$

The last five constraints are known as the complementarity conditions. The entire model is known as a linear complementarity problem.

Rather than use a specialized linear complementarity algorithm, we will simply use the integer programming capabilities for LINGO to model the problem as follows:

```
MODEL: ! Two person nonconstant sum game.(BIMATRX);
 SETS:
   OPTA: PA, SLKA, NOTUA, COSA;
   OPTB: PB, SLKB, NOTUB, COSB;
   BXA( OPTB, OPTA): C2A, C2B;
 ENDSETS
 DATA:
 OPTB = BNAD BYAD;
 OPTA = ANAD AMAD AHAD;
  C2A =  2    3    1  ! C2A( I, J) = cost to A if B;
         5    4    5; ! chooses row I, A chooses col J;
  C2B =  2    4    7  ! C2B( I, J) = cost to B if B;
         1    5    6; ! chooses row I, A chooses col J;
 ENDDATA
!-----------------------------------------------;
! Conditions for A, for each option J;
 @FOR( OPTA( J):
! Set CBSTA= cost of strategy J, if J is used by A;
   CBSTA = COSA( J) - SLKA( J);
   COSA( J) = @SUM( OPTB( I): C2A( I, J) * PB( I));
! Force SLKA( J) = 0 if strategy J is used;
   SLKA( J) <= NOTUA( J) * @MAX( OPTB( I):
     C2A( I, J));
! NOTUA( J) = 1 if strategy J is not used;
   PA( J) <= 1 - NOTUA( J);
! Either strategy J is used or it is not used;
   @BIN( NOTUA( J));
       );
! A must make a decision;
 @SUM( OPTA( J): PA( J)) = 1;
! Conditions for B;
 @FOR( OPTB( I):
! Set CBSTB = cost of strategy I, if I is used by
      B;
   CBSTB =  COSB( I)  - SLKB( I);
   COSB( I) = @SUM( OPTA( J): C2B( I, J) * PA( J));
! Force SLKB( I) = 0 if strategy I is used;
   SLKB( I) <= NOTUB( I) * @MAX( OPTA( J):
      C2B( I, J));
! NOTUB( I) = 1 if strategy I is not used;
   PB( I) <= 1 - NOTUB( I);
! Either strategy I is used or it is not used;
   @BIN( NOTUB( I));
       );
!  B must make a decision;
 @SUM( OPTB( I): PB( I)) = 1;
  END
```

A solution is:

```
       Variable           Value
          CBSTA        3.666667
          CBSTB        5.500000
      PA( AMAD)       0.5000000
      PA( AHAD)       0.5000000
    SLKA( ANAD)       0.3333333
   NOTUA( ANAD)        1.000000
    COSA( ANAD)        4.000000
    COSA( AMAD)        3.666667
    COSA( AHAD)        3.666667
      PB( BNAD)       0.3333333
      PB( BYAD)       0.6666667
    COSB( BNAD)        5.500000
    COSB( BYAD)        5.500000
```

The solution indicates that firm *A* should not use option 1(No ads) and should randomly choose with equal probability between options 2 and 3. Firm *B* should choose its option 2(Advertise) twice as frequently as it chooses its option 1(Do not advertise).

The objective function value, reduced costs and dual prices can be disregarded. Using our original loss table, we can calculate the following:

Situation			**Weighted Contribution To Total Loss of**	
A	**B**	**Probability**	**A**	**B**
No Ads	No Ads	$0 \times 1/3$	0	0
No Ads	Ads	$0 \times 2/3$	0	0
Advertise Medium	No Ads	$1/2 \times 1/3$	$(1/6) \times (-3)$	$(1/6) \times (-2)$
Advertise Medium	Ads	$1/2 \times 2/3$	$(1/3) \times (-2)$	$(1/3) \times (-1)$
Advertise High	No Ads	$1/2 \times 1/3$	$(1/6) \times (-5)$	$(1/6) \times (1)$
Advertise	Ads	$1/2 \times 2/3$	$(1/3) \times (-1)$	$(1/3) \times (0)$
			–2.3333	–0.5

Thus, following the randomized strategy suggested, *A* would have an expected profit of 2.33 million; whereas, *B* would have an expected profit of 0.5 million. Contrast this with the fact that, if *A* and *B* cooperated, they could each have an expected profit of 4 million.

16.3.3 Bimatrix Games with Several Solutions

When a nonconstant sum game has multiple or alternative stable solutions, life gets more complicated. The essential observation is we must look outside our narrow definition of "stable solution" to decide which of the stable solutions, if any, would be selected in reality.

Consider the following nonconstant sum two-person game:

Figure 16.3 Bimatrix Games

		Firm A 1	Firm A 2
Firm B	1	200 / 200	10 / 160
Firm B	2	160 / 10	100 / 100

As before, the numbers represent losses.

First, observe the one solution that is stable according to our definition: (*I*) Firm *A* always chooses option 1 and Firm *B* always chooses option 2. Firm *A* is not motivated to switch to 2 because its losses would increase to 100 from 10. Similarly, *B* would not switch to 1 from 2 because its losses would increase to 200 from 160. The game is symmetric in the players, so similar arguments apply to the solution (*II*): *B* always chooses 1 and *A* always chooses 2.

Which solution would result in reality? It probably depends upon such things as the relative wealth of the two firms. Suppose:

i. *A* is the wealthier firm,
ii. the game is repeated week after week, and
iii. currently the firms are using solution *II*.

After some very elementary analysis, *A* concludes it much prefers solution *I*. To move things in this direction, *A* switches to option 1. Now, it becomes what applied mathematicians call a game of "chicken". Both players are taking punishment at the rate of 200 per week. Either player could improve its lot by 200 − 160 = 40 by unilaterally switching to its option 2. However, its lot would be improved a lot more (i.e., 200 − 10 = 190) if its opponent unilaterally switched. At this point, a rational *B* would probably take a glance at *A*'s balance sheet and decide *B* switching to option 2 is not such a bad decision. When a game theory problem has multiple solutions, any given player would like to choose that stable solution which is best for it. If the player has the wherewithal to force such a solution (e.g., because of its financial size), then this solution is sometimes called a Stackelberg equilibrium.

If it is not clear which firm is wealthier, then the two firms may decide a cooperative solution is best (e.g., alternate between solutions *I* and *II* in alternate weeks). At this point, however, federal antitrust authorities might express a keen interest in this bimatrix game.

We conclude a "stable" solution is stable only in a local sense. When there are multiple stable solutions, we should really look at all of them and take into account other considerations in addition to the loss matrix.

16.4 Nonconstant-Sum Games Involving Two or More Players

The most unrealistic assumption underlying classical two-person constant-sum game theory is the sum of the payoffs to all players must sum to zero (actually a constant, without loss of generality). In reality, the total benefits are almost never constant. Usually, total benefits increase if the players cooperate, so these situations are sometimes called cooperative games. In these nonconstant-sum games, the difficulty then becomes one of deciding how these additional benefits due to cooperation should be distributed among the players.

There are two styles for analyzing nonconstant sum games. If we restrict ourselves to two persons, then so-called bimatrix game theory extends the methods for two-person constant sum games to nonconstant sum games. If there are three or more players, then *n-person game theory* can be used in selecting a decision strategy. The following example illustrates the essential concepts of n-person game theory.

Three property owners, A, B, and C, own adjacent lakefront property on a large lake. A piece of property on a large lake has higher value if it is protected from wave action by a seawall. A, B, and C are each considering building a seawall on their properties. A seawall is cheaper to build on a given piece of property if either or both of the neighbors have seawalls. For our example, A and C already have expensive buildings on their properties. B does not have buildings and separates A from C (i.e., B is between A and C). The net benefits of a seawall for the three owners are summarized below:

Owners Who Cooperate, i.e., Build While Others Do Not	Net Benefit to Cooperating Owners
A alone	1.2
B alone	0
C alone	1
A and B	4
A and C	3
B and C	4
A, B, and C	7

Obviously, all three owners should cooperate and build a unified seawall because then their total benefits will be maximized. It appears B should be compensated in some manner because he has no motivation to build a seawall by himself. Linear programming can provide some help in selecting an acceptable allocation of benefits.

Denote by v_A, v_B, and v_C the net benefits, which are to be allocated to owners A, B, and C. No owner or set of owners will accept an allocation that is less than that, which they would enjoy if they acted alone. Thus, we can conclude:

$$v_A \geq 1.2$$
$$v_B \geq 0$$
$$v_C \geq 1$$
$$v_A + v_B \geq 4$$
$$v_A + v_C \geq 3$$
$$v_B + v_C \geq 4$$
$$v_A + v_B + v_C \leq 7$$

That is, any allocation satisfying the above constraints should be self-enforcing. No owner would be motivated to not cooperate. He cannot do better by himself. The above constraints describe what is called the "core" of the game. Any solution (e.g., $v_A = 3$, $v_B = 1$, $v_C = 3$) satisfying these constraints is said to be in the core.

Various objective functions might be appended to this set of constraints to give an LP. The objective could take into account secondary considerations. For example, we might choose to maximize the minimum benefit. The LP in this case is:

Maximize z
subject to $z \leq v_A$; $z \leq v_B$; $z \leq v_C$

$$v_A \geq 1.2$$
$$v_C \geq 1$$
$$v_A + v_B \geq 4$$
$$v_A + v_C \geq 3$$
$$v_A + v_B + v_C \leq 7$$

A solution is $v_A = v_B = v_C = 2.3333$.

Note the core can be empty. That is, there is no feasible solution. This would be true, for example, if the value of the coalition A, B, C was 5.4 rather than 7. This situation is rather interesting. Total benefits are maximized by everyone cooperating. However, total cooperation is inherently unstable when benefits are 5.4. There will always be a pair of players who find it advantageous to form a subcoalition and improve their benefits (at the considerable expense of the player left out). As an example, suppose the allocations to A, B, and C under full cooperation are 1.2, 2.1, and 2.1, respectively. At this point, A would suggest to B the two of them exclude C and cooperate between the two of them. A would suggest to B the allocation of 1.8, 2.2, and 1. This is consistent with the fact that A and B can achieve a total of 4 when cooperating. At this point, C might suggest to A that the two of them cooperate and thereby select an allocation of 1.9, 0, 1.1. This is inconsistent with the fact that A and C can achieve a total of 3 when cooperating. At this point, B suggests to C etc. Thus, when the core is empty, it may be everyone agrees that full cooperation can be better for everyone. There nevertheless must be an enforcement mechanism to prevent "greedy" members from pulling out of the coalition.

16.4.1 Shapley Value

Another popular allocation method for cooperative games is the Shapley Value. The rule for the Shapley Value allocation is that each player should be awarded his average marginal contribution to the coalition if one considers all possible sequences for forming the full coalition. The following table illustrates for the previous example:

	Marginal value of player		
Sequence	A	B	C
A B C	1.2	2.8	3
A C B	1.2	4	1.8
B A C	4	0	3
B C A	3	0	4
C A B	2	4	1
C B A	3	3	1
Total:	14.4	13.8	13.8
Average:	2.4	2.3	2.3

Thus, the Shapley value allocates slightly more, 2.4, to Player *A* in our example. For this example, as with most typical cooperative games, the Shapley Value allocation is in the core if the core is non-empty.

16.5 Problems

1. Both Big Blue, Inc. and Golden Apple, Inc. are "market oriented" companies and feel market share is everything. The two of them have 100% of the market for a certain industrial product. Blue and Gold are now planning the marketing campaigns for the upcoming selling season. Each company has three alternative marketing strategies available for the season. Gold's market share as a function of both the Blue and Gold decisions are tabulated below:

Payment To Blue by Gold as a Function of Both Decisions

		Blue Decision		
		A	**B**	**C**
	X	.4	.8	.6
Gold Decision	**Y**	.3	.7	.4
	Z	.5	.9	.5

Both Blue and Gold know the above matrix applies. Each must make their decision before learning the decision of the other. There are no other considerations.

a) What decision do you recommend for Gold?

b) What decision do you recommend for Blue?

2. Formulate an LP for finding the optimal policies for Blue and Gold when confronted with the following game:

Payment To Blue By Gold as a Function of Both Decisions

		Blue Decision			
		A	**B**	**C**	**D**
Gold Decision	**X**	2	–2	1	6
	Y	–1	4	5	–1

3. Two competing manufacturing firms are contemplating their advertising options for the upcoming season. The profits for each firm as a function of the actions of both firms are shown below. Both firms know this table:

Profit Contributions

		Fulcher Fasteners		
		Option A	**Option B**	**Option C**
Repicky Rivets	**Option Y**	4 10	8 4	6 6
	Option X	8 8	12 2	10 4

a) Which pair of actions is most profitable for the pair?

b) Which pairs of actions are stable?

c) Presuming side payments are legal, how much would which firm have to pay the other firm in order to convince them to stick with the most profitable pair of actions?

4. The three neighboring communities of Parched, Cactus and Tombstone are located in the desert and are analyzing their options for improving their water supplies. An aqueduct to the mountains would satisfy all their needs and cost in total $730,000. Alternatively, Parched and Cactus could dig and share an artesian well of sufficient capacity, which would cost $580,000. A similar option for Cactus and Tombstone would cost $500,000. Parched, Cactus and Tombstone could each individually distribute shallow wells over their respective surface areas to satisfy their needs for respective costs of $300,000, $350,000 and $250,000.

   Formulate and solve a simple LP for finding a plausible way of allocating the $730,000 cost of an aqueduct among the three communities.

5. Sportscasters say Team *I* is out of the running if the number of games already won by *I* plus the number of remaining games for Team *I* is less than the games already won by the league leader. It is frequently the case that a team is mathematically out of the running even before that point is reached. By Team *I* being mathematically out of the running, we mean there is no combination of wins and losses for the remaining games in the season such that Team *I* could end the season having won more games than any other team. A third-place team might find itself mathematically though not obviously out of the running if the first and second place teams have all their remaining games against each other.

   Formulate a linear program that will not have a feasible solution if Team *I* is no longer in the running.

The following variables may be of interest:

x_{jk} = number of times Team *j* may beat Team *k* in the season's remaining games and Team *I* still win more games than anyone else.

The following constants should be used:

R_{jk} = number of remaining games between Team *j* and Team *k*. Note the number of times *j* beats *k* plus the number of times *k* beats *j* must equal R_{jk}.
T_k = total number of games won by Team *k* to date. Thus, the number of games won at season's end by Team *k* is T_k plus the number of times it beat other teams.

6. In the 1983 NBA basketball draft, two teams were tied for having the first draft pick, the reason being that they had equally dismal records the previous year. The tie was resolved by two flips of a coin. Houston was given the opportunity to call the first flip. Houston called it correctly and therefore was eligible to call the second flip. Houston also called the second flip correctly and thereby won the right to negotiate with the top-ranked college star, Ralph Sampson. Suppose you are in a similar two-flip situation. You suspect the special coin used may be biased, but you have no idea which way. If you are given the opportunity to call the first flip, should you definitely accept, be indifferent, or definitely reject the opportunity (and let the other team call the first flip). State your assumptions explicitly.

7. A recent auction for a farm described it as consisting of two tracts as follows:

   Tract 1: 40 acres, all tillable, good drainage.
   Tract 2: 35 acres, of which 30 acres are tillable, 5 acres containing pasture, drainage ditch and small pond.

   The format of the auction was described as follows. First Tract 1 and Tract 2 will each be auctioned individually. Upon completion of bidding on Tract 1 and Tract 2, there will be a 15 minute intermission. After that period of time, this property will be put together as one tract of farmland. There will be a premium added to the total dollar price of Tract 1 and Tract 2. This total dollar amount will be the starting price of the 75 acres. If, at that time, no one bids, then the property will go to the highest bidders on Tracts 1 and 2. Otherwise, if the bid increases, then it will be sold as one.

   Can you think of some modest changes in the auction procedure that might increase the total amount received for the seller? What are some of the game theory issues facing the individual bidders in this case?