13

Portfolio Optimization

13.1 Introduction

Portfolio models are concerned with investment where there are typically two criteria: expected return and risk. The investor wants the former to be high and the latter to be low. There is a variety of measures of risk. The most popular measure of risk has been variance in return. Even though there are some problems with it, we will first look at it very closely. All the nontrivial LINGO models shown here can be downloaded from www.lindo.com, in the MODELS library.

13.2 The Markowitz Mean/Variance Portfolio Model

The portfolio model introduced by Markowitz (1959), see also Roy (1952), assumes an investor has two considerations when constructing an investment portfolio: expected return and variance in return (i.e., risk). Variance measures the variability in realized return around the expected return, giving equal weight to realizations below the expected and above the expected return. The Markowitz model might be mildly criticized in this regard because the typical investor is probably concerned only with variability below the expected return, so-called downside risk. The Markowitz model requires two major kinds of information: (1) the estimated expected return for each candidate investment and (2) the covariance matrix of returns. The covariance matrix characterizes not only the individual variability of the return on each investment, but also how each investment's return tends to move with other investments. We assume the reader is familiar with the concepts of variance and covariance as described in most intermediate statistics texts. Part of the appeal of the Markowitz model is it can be solved by efficient quadratic programming methods. Quadratic programming is the name applied to the class of models in which the objective function is a quadratic function and the constraints are linear. Thus, the objective function is allowed to have terms that are products of two variables such as x^2 and $x \times y$.

Quadratic programming is computationally appealing because the algorithms for linear programs can be applied to quadratic programming with only modest modifications. Loosely speaking, the reason only modest modification is required is the first derivative of a quadratic function is a linear function. Because LINGO has a general nonlinear solver, the limitation to quadratic functions is helpful, but not crucial.

13.2.1 Example

We will use some publicly available data from Markowitz (1959). Eppen, Gould and Schmidt (1991) use the same data. The following table shows the increase in price, including dividends, for three stocks over a twelve-year period:

		Growth in		
Year	**S&P500**	**ATT**	**GMC**	**USX**
43	1.259	1.300	1.225	1.149
44	1.198	1.103	1.290	1.260
45	1.364	1.216	1.216	1.419
46	0.919	0.954	0.728	0.922
47	1.057	0.929	1.144	1.169
48	1.055	1.056	1.107	0.965
49	1.188	1.038	1.321	1.133
50	1.317	1.089	1.305	1.732
51	1.240	1.090	1.195	1.021
52	1.184	1.083	1.390	1.131
53	0.990	1.035	0.928	1.006
54	1.526	1.176	1.715	1.908

For reference later, we have also included the change each year in the Standard and Poor's/S&P 500 stock index. To illustrate, in the first year, *ATT* appreciated in value by 30%. In the second year, *GMC* appreciated in value by 29%. Based on the twelve years of data, we can use any standard statistical package to calculate a covariance matrix for three stocks: *ATT*, *GMC*, and *USX*. The matrix is:

	ATT	**GMC**	**USX**
ATT	0.01080754	0.01240721	0.01307513
GMC	0.01240721	0.05839170	0.05542639
USX	0.01307513	0.05542639	0.09422681

From the same data, we estimate the expected return per year, including dividends, for *ATT*, *GMC*, and *USX* as 0.0890833, 0.213667, and 0.234583, respectively.

The correlation matrix makes it more obvious how two random variables move together. The correlation between two random variables equals the covariance between the two variables, divided by the product of the standard deviations of the two random variables. For our three investments, the correlation matrix is as follows:

	ATT	**GMC**	**USX**
ATT	1.0		
GMC	0.493895589	1.0	
USX	0.409727718	0.747229121	1.0

The correlation can be between −1 and +1 with +1 being a high correlation between the two. Notice *GMC* and *USX* are highly correlated. *ATT* tends to move with *GMC* and *USX*, but not nearly so much as *GMC* moves with *USX*.

Let the symbols *ATT*, *GMC*, and *USX* represent the fraction of the portfolio devoted to each of the three stocks. Suppose, we desire a 15% yearly return. The entire model can be written as:

```
MODEL:
  !Minimize end-of-period variance in portfolio value;
    [VAR] MIN = .01080754 * ATT * ATT +.01240721 * ATT * GMC + .01307513
    * ATT * USX +.01240721 * GMC * ATT +.05839170 * GMC * GMC +.05542639
    * GMC * USX +.01307513 * USX * ATT +.05542639 * USX * GMC +.09422681
    * USX * USX;
  ! Use exactly 100% of the starting budget;
    [BUD] ATT + GMC + USX = 1;
  ! Required wealth at end of period;
    [RET] 1.089083 * ATT + 1.213667 * GMC + 1.234583 * USX >= 1.15;
END
```

Note the two constraints are effectively in the same units. The first constraint is effectively a "beginning inventory" constraint, while the second constraint is an "ending inventory" constraint. We could have stated the expected return constraint just as easily as:

```
.0890833 * ATT + .213667 * GMC + .234583 * USX >= .15
```

Although perfectly correct, this latter style does not measure end-of-period state in quite the same way as start-of-period state. Fans of consistency may prefer the former style.

The equivalent sets-based formulation of the model follows:

```
MODEL:
 SETS:
  ASSET: AMT, RET;
  COVMAT(ASSET, ASSET): VARIANCE;
 ENDSETS
 DATA:
  ASSET =      ATT      GMC       USX;
!Covariance matrix and expected returns;
  VARIANCE = .01080754 .01240721 .01307513
             .01240721 .05839170 .05542639
             .01307513 .05542639 .09422681;
       RET = 1.0890833  1.213667  1.234583;
    TARGET = 1.15;
 ENDDATA
! Minimize the end-of-period variance in portfolio value;
[VAR] MIN = @SUM( COVMAT(I, J): AMT(I) * AMT(J) * VARIANCE(I, J));
!  Use exactly 100% of the starting budget;
[BUDGET] @SUM( ASSET: AMT) = 1;
! Required wealth at end of period;
[RETURN] @SUM( ASSET: AMT * RET) >= TARGET;
END
```

When we solve the model, we get:

```
Optimal solution found at step:          4
Objective value:                0.2241375E-01
 Variable           Value        Reduced Cost
   TARGET        1.150000           0.0000000
AMT( ATT)       0.5300926           0.0000000
AMT( GMC)       0.3564106           0.0000000
AMT( USX)       0.1134968           0.0000000
RET( ATT)        1.089083           0.0000000
RET( GMC)        1.213667           0.0000000
RET( USX)        1.234583           0.0000000
      Row    Slack or Surplus      Dual Price
      VAR       0.2241375E-01        1.000000
   BUDGET       0.0000000           0.3621387
   RETURN       0.0000000          -0.3538836
```

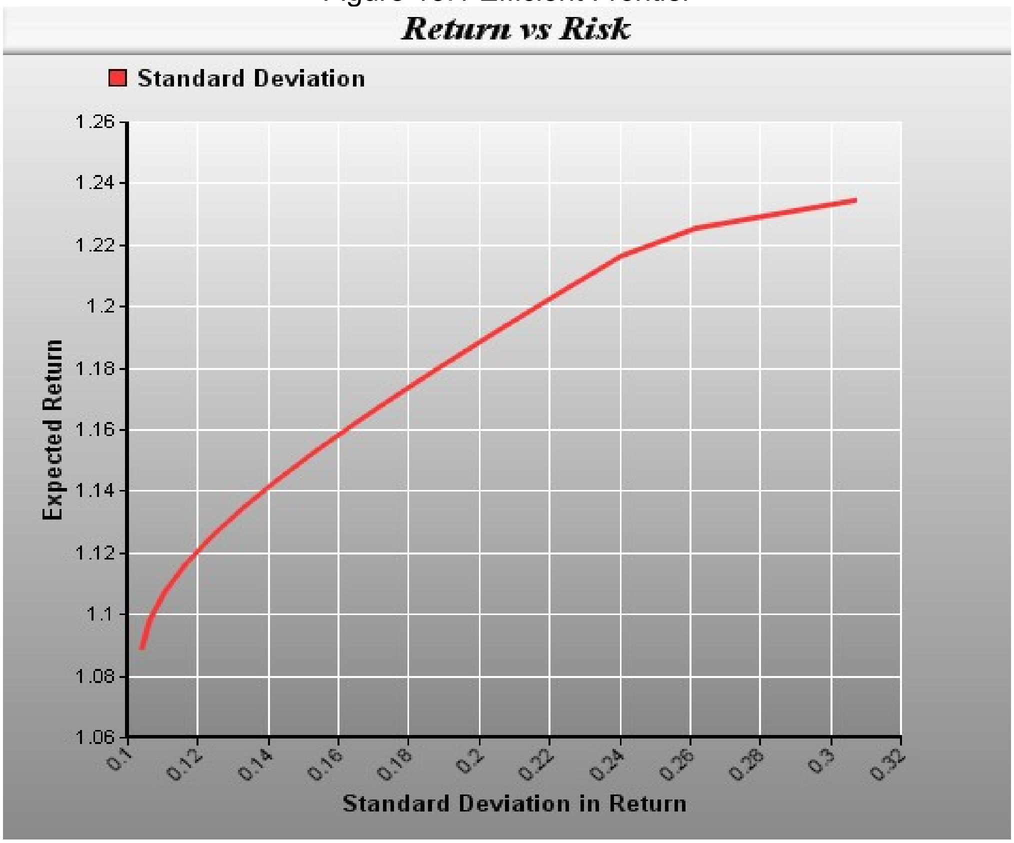
The solution recommends about 53% of the portfolio be put in *ATT*, about 36% in *GMC* and just over 11% in *USX*. The expected return is 15%, with a variance of 0.02241381 or, equivalently, a standard deviation of about 0.1497123.

We based the model simply on straightforward statistical data based on yearly returns. In practice, it may be more typical to use monthly rather than yearly data as a basis for calculating a covariance. Also, rather than use historical data for estimating the expected return of an asset, a decision maker might base the expected return estimate on more current, proprietary information about expected future performance of the asset. One may also wish to use considerable care in estimating the covariances and the expected returns. For example, one could use quite recent data to estimate the standard deviations. A large set of data extending further back in time could be used to estimate the correlation matrix. Then, using the relationship between the correlation matrix and the covariance matrix, one could derive a covariance matrix.

13.3 Dualing Objectives: Efficient Frontier and Parametric Analysis

There is no precise way for an investor to determine the "correct" tradeoff between risk and return. Thus, one is frequently interested in looking at the tradeoff between the two. If an investor wants a higher expected return, she generally has to "pay for it" with higher risk. In finance terminology, we would like to trace out the efficient frontier of return and risk. If we solve for the minimum variance portfolio over a range of values for the expected return, ranging from 0.0890833 to 0.234583, we get the following plot or tradeoff curve for our little three-asset example:

Figure 13.1 Efficient Frontier



Notice the "knee" in the curve as the required expected return increases past 1.21894. This is the point where ATT drops out of the portfolio. This graph was generated using model PortEfFront12.lng

13.3.1 Portfolios with a Risk-Free Asset

When one of the investments available is risk free, then the optimal portfolio composition has a particularly simple form. Suppose the opportunity to invest money risk free (e.g., in government treasury bills) at 5% per year has just become available. Working with our previous example, we now have a fourth investment instrument that has zero variance and zero covariance. There is no limit on how much can be invested at 5%. We ask the question: How does the portfolio composition change as the desired rate of return changes from 15% to 5%?

We will use the following slight generalization of the original Markowitz example model. Notice a fourth instrument, treasury bills (*TBILL*), has been added:

```
MODEL:
  ! Add a riskless asset, TBILL;
  ! Minimize end-of-period variance in portfolio value;
    [VAR] MIN = .01080754* ATT * ATT +.01240721* ATT * GMC  +.01307513*
    ATT * USX +.01240721* GMC * ATT +.05839170* GMC * GMC +.05542639*
    GMC * USX +.01307513* USX * ATT +.05542639* USX * GMC +.09422681*
    USX * USX;
  !  Use exactly 100% of the starting budget;
    [BUD]  ATT + GMC + USX + TBILL = 1;
  ! Required wealth at end of period;
    [RET] 1.089083 * ATT + 1.213667 * GMC + 1.234583 * USX + 1.05 *
    TBILL >= 1.15;
END
```

Alternatively, this can be also modeled using the sets formulation:

```
MODEL:
  SETS:
   ASSET: AMT, RET;
   COVMAT(ASSET, ASSET): VARIANCE;
  ENDSETS
  DATA:
       ASSET=      ATT,     GMC,     USX,   TBILL;
  !Covariance matrix;
    VARIANCE = .01080754 .01240721 .01307513   0
               .01240721 .05839170 .05542639   0
               .01307513 .05542639 .09422681   0
                0         0         0          0;
         RET = 1.0890833 1.213667 1.234583, 1.05;
         TARGET = 1.15;
  ENDDATA
  ! Minimize the end-of-period variance in portfolio value;
  [VAR] MIN= @SUM( COVMAT( I, J): AMT( I)* AMT( J) * VARIANCE( I, J));
  !  Use exactly 100% of the starting budget;
  [BUDGET] @SUM(ASSET: AMT) = 1;
  ! Required wealth at end of period;
  [RETURN] @SUM( ASSET: AMT * RET) >= TARGET;
END
```

When solved, we find:

```
Optimal solution found at step:         8
Objective value:                0.2080344E-01
Variable           Value        Reduced Cost
     ATT       0.8686550E-01      -0.2093725E-07
     GMC       0.4285285           0.0000000
     USX       0.1433992          -0.2218303E-07
   TBILL       0.3412068           0.0000000
     Row    Slack or Surplus      Dual Price
     VAR       0.2080344E-01        1.000000
     BUD       0.0000000           0.4368723
     RET       0.0000000          -0.4160689
```

Notice more than 34% of the portfolio was invested in the risk-free investment, even though its return rate, 5%, is less than the target of 15%. Further, the variance has dropped to about 0.0208 from about 0.0224.

What happens as we decrease the target return towards 5%? Clearly, at 5%, we would put zero in *ATT*, *GMC*, and *USX*. A simple form of solution would be to keep the same proportions in *ATT*, *GMC*, and *USX*, but just change the allocation between the risk-free asset and the risky ones. Let us check an intermediate point. When we decrease the required return to 10%, we get the following solution:

```
Optimal solution found at step:         8
Objective value:                0.5200865E-02
Variable           Value        Reduced Cost
     ATT       0.4342898E-01       0.0000000
     GMC       0.2142677           0.2857124E-06
     USX       0.7169748E-01       0.1232479E-06
   TBILL       0.6706058           0.0000000
     Row    Slack or Surplus      Dual Price
     VAR       0.5200865E-02        1.000000
     BUD       0.0000000           0.2184348
     RET       0.2384186E-07      -0.2080331
```

This solution supports our conjecture:

> *as we change our required return, the relative proportions devoted to risky investments do not change. Only the allocation between the risk-free asset and the risky assets change.*

From the above solution, we observe that, except for round-off error, the amount invested in *ATT*, *GMC*, and *USX* is allocated in the same way for both solutions. Thus, two investors with different risk preferences would nevertheless both carry the same mix of risky stocks in their portfolio. Their portfolios would differ only in the proportion devoted to the risk-free asset. Our observation from the above example in fact holds in general. Thus, the decision of how to allocate funds among stocks, given the amount to be invested, can be separated from the questions of risk preference. Tobin received the Nobel Prize in 1981, largely for noticing the above feature, the so-called Separation Theorem. So, if you noticed it, you must be Nobel Prize caliber.

13.3.2 The Sharpe Ratio

For some portfolio *p*, of risky assets, excluding the risk-free asset, let:

R_p = its expected return,
s_p = its standard deviation in return, and
r_0 = the return of the risk-free asset.

A plausible single measure (as opposed to the two measures, risk and return) of attractiveness of portfolio *p* is the Sharpe ratio:

$$(R_p - r_0) / s_p$$

In words, it measures how much additional return we achieved for the additional risk we took on, relative to putting all our money in the risk-free asset.

It happens the portfolio that maximizes this ratio has a certain well-defined appeal. Suppose:

t = our desired target return,
w_p = fraction of our wealth we place in portfolio *p* (the rest placed in the risk-free asset).

To meet our return target, we must have:

$$(1 - w_p) * r_0 + w_p * R_p = t.$$

The standard deviation of our total investment is:

$$w_p * s_p.$$

Solving for w_p in the return constraint, we get:

$$w_p = (t - r_0) / (R_p - r_0).$$

Thus, the standard deviation of the portfolio is:

$$w_p * s_p = [(t - r_0) / (R_p - r_0)] * s_p.$$

Minimizing the portfolio standard deviation means:

$$\text{Min } [(t - r_0) / (R_p - r_0)] * s_p$$

or

$$\text{Min } [(t - r_0) * s_p / (R_p - r_0)].$$

This is equivalent to:

$$\text{Max } (R_p - r_0) / s_p.$$

So, regardless of our risk/return preference, the money we invest in risky assets should be invested in the risky portfolio that maximizes the Sharpe ratio.

The following illustrates for when the risk free rate is 5%:

```
MODEL:
! Maximize the Sharpe ratio;
 MAX =
(1.089083*ATT + 1.213667*GMC + 1.234583*USX - 1.05)/
 ((.01080754 * ATT * ATT   + .01240721 * ATT * GMC
 + .01307513 * ATT * USX + .01240721 * GMC * ATT
 + .05839170 * GMC * GMC + .05542639 * GMC * USX
 + .01307513 * USX * ATT + .05542639 * USX * GMC
 + .09422681 * USX * USX)^.5);
! Use exactly 100% of the starting budget;
 [BUD]  ATT + GMC + USX = 1;
END
```

The solution is:

```
Optimal solution found at step:        7
Objective value:                0.6933179
Variable           Value        Reduced Cost
     ATT       0.1319260          0.1263448E-04
     GMC       0.6503984          0.0000000
     USX       0.2176757          0.1250699E-04
```

Notice the relative proportions of *ATT*, *GMC*, and *USX* are the same as in the previous model where we explicitly included a risk free asset with a return of 5%. For example, notice that, except for round-off error:

```
.1319262/ .6503983 = 0.08686515/ .4285286.
```

13.4 Important Variations of the Portfolio Model

There are several issues that may concern you when you think about applying the Markowitz model in its simple form:

a) As we increase the number of assets to consider, the size of the covariance matrix becomes overwhelming. For example, 1000 assets implies 1,000,000 covariance terms, or at least 500,000 if symmetry is exploited.
b) If the model were applied every time new data become available (e.g., weekly), we would "rebalance" the portfolio frequently, making small, possibly unimportant adjustments in the portfolio.
c) There are no upper bounds on how much can be held of each asset. In practice, there might be legal or regulatory reasons for restricting the amount of any one asset to no more than, say, 5% of the total portfolio. Some portfolio managers may set the upper limit on a stock to one day's trading volume for the stock. The reasoning being, if the manager wants to "unload" the stock quickly, the market price would be affected significantly by selling so much.

Two approaches for simplifying the covariance structure have been proposed: the scenario approach and the factor approach. For the issue of portfolio "nervousness", the incorporation of transaction costs is useful.

13.4.1 Portfolios with Transaction Costs

The models above do not tell us much about how frequently to adjust our portfolio as new information becomes available (i.e., new estimates of expected return and new estimates of variance). If we applied the above models every time new information became available, we would be constantly adjusting our portfolio. This might make our broker happy because of all the commission fees, but that should be a secondary objective at best. The important observation is that there are costs associated with buying and selling. There are the obvious commission costs, and the not so obvious bid-ask spread. The bid-ask spread is effectively a transaction cost for buying and selling.

The method we will describe assumes transaction costs are paid at the beginning of the period. It is a straightforward exercise to modify the model to handle the case of transaction costs paid at the end of the period. The major modifications to the basic portfolio model are:

a) We must introduce two additional variables for each asset, an "amount bought" variable and an "amount sold" variable.
b) The budget constraint must be modified to include money spent on commissions.
c) An additional constraint must be included for each asset to enforce the requirement:

    amount invested in asset i = (initial holding of i) + (amount bought of i) − (amount sold of i).

Example

Suppose we have to pay a 1% transaction fee on the amount bought or sold of any stock and our current portfolio is 50% *ATT*, 35% *GMC*, and 15% *USX*. This is pretty close to the optimal mix. Should we incur the cost of adjusting? The following is the relevant model:

```
MODEL:
  [VAR] MIN = .01080754 * ATT * ATT +.01240721 * ATT * GMC +.01307513 *
    ATT * USX +.01240721 * GMC * ATT +.05839170 * GMC * GMC +.05542639
    * GMC * USX +.01307513 * USX * ATT +.05542639 * USX * GMC +.09422681
    * USX * USX;
  [BUD] ATT + GMC + USX + .01 * ( BA + BG + BU + SA + SG + SU) = 1;
  [RET] 1.089083 * ATT + 1.213667 * GMC + 1.234583 * USX >= 1.15;
  [NETA] ATT = .50 + BA - SA;
  [NETG] GMC = .35 + BG - SG;
  [NETU] USX = .15 + BU - SU;
END
```

The *BUD* constraint says the total uses of funds must equal 1. Another way of interpreting the *BUD* constraint is to subtract each of the NET constraints from it. We then get:

```
[BUD].01 * (BA + BG + BU + SA + SG + SU) + BA + BG + BU=SA + SG + SU;
```

It says any purchases plus transaction fees must be funded by selling.

The solution follows:

```
Optimal solution found at step:         4
Objective value:                0.2261146E-01
Variable           Value        Reduced Cost
     ATT        0.5264748          0.0000000
     GMC        0.3500000          0.0000000
     USX        0.1229903          0.0000000       .9994651
      BA        0.2647484E-01      0.0000000
      BG        0.0000000          0.4824887E-02
      BU        0.0000000          0.6370753E-02
      SA        0.0000000          0.6370753E-02
      SG        0.0000000          0.1545865E-02
      SU        0.2700968E-01      0.0000000
```

The solution recommends buying a little bit more *ATT*, neither buy nor sell any *GMC*, and sell a little *USX*.

For reference, the following is the sets formulation of the above model:

```
MODEL:
  SETS:
     ASSET: AMT, RETURN, BUY, SELL, START;
     COVMAT( ASSET, ASSET):VARIANCE;
  ENDSETS
  DATA:
    ASSET =      ATT,      GMC,      USX;
   VARIANCE = .0108075 .0124072 .0130751
              .0124072 .0583917 .0554264
              .0130751 .0554264 .0942268;
     RETURN = 1.089083 1.213667 1.234583;
      START = .5   .35 .15;
     TARGET = 1.15;
  ENDDATA
    [VAR] MIN = @SUM( COVMAT(I, J): AMT(I) * AMT(J) * VARIANCE(I, J));
    [BUD] @SUM( ASSET(I): AMT(I) + .01 * ( BUY(I) + SELL(I))) = 1;
    [RET] @SUM( ASSET: AMT * RETURN) >= TARGET;
    @FOR( ASSET(I): [NET] AMT(I) = START(I) + BUY(I) - SELL(I););
END
```

13.4.2 Nonlinear Transaction Costs

If we look more closely at transaction costs, we will probably find that they are nonlinear. There may be: *i*) a volume-independent fixed cost of doing a transaction, and *ii*) a market impact cost. The latter corresponds to the effect that if we try to buy a lot of something it will tend to drive up the price, and if we try to sell a lot of something it will tend to drive down the price. In this section, we will consider only market impact costs. If we buy an amount B_j of stock j, one representation of the cost per dollar purchased of stock j is:

$c_j*B_j + m_j*B_j^{\,p}$, where $1 < p$, and

c_j = proportional transaction cost. E.g., if the bid and ask prices are 99.5 and 100.5, we might state the list price as 100. If the commission rate is 0.0025, we would then set $c_j = 0.5/100 + 0.0025 = 0.0075$.

m_j = the market impact coefficient. We expect this coefficient to be inversely related to the daily trading volume of stock j, and directly related to the average daily price spread. E.g., if the average trading volume is high, then our trade will tend to not have a big impact on price.

Example: Below is an model, PortMimpact.lng, that is an extension of our previous transaction cost example, but with a market impact term in the transaction cost computation.

```
MODEL:
! Portfolio model with a market impact component in
computeing transaction costs;         !(PortMimpact.lng);
SETS:
 ASSET: AMT, RETURN, STD, BUY, SELL, START, C, M, BOS;
 TMAT( ASSET, ASSET) | &1 #GE# &2: CORR;
ENDSETS
DATA:
 ASSET =   ATT          GMC         USX       TBILL;
 RETURN= 1.089083    1.213667    1.234583  1.00;
  START=    0.5         0.35        0.15      0.0;
! Proportional transaction costs;
    C  =  0.01        0.01        0.01      0.005;
! Market impact coefficients;
    M  =  0.008        0.009        0.001      0.0;
   POW = 1.5;  ! Power to use in approximating market impact;
! Standard deviation in return;
   STD = 0.10395932  0.24164375  0.30696386  0;

! Correlation matrix;
  CORR = 1
         0.493895589 1
         0.409727718 0.747229121 1
         0           0           0           1;

 TARGET = 1.15; ! Target growth factor;
ENDDATA
! Minimize variance of portfolio;
   MIN = (@SUM( ASSET( I): STD( I)*STD(I)*AMT( I)^2) +
      2 * @SUM( TMAT( I, J) | I #GT# J:
                 AMT( I) * AMT( J)* CORR( I, J) *( STD( I) * STD( J)))) ;
@FOR( ASSET(I):
! Post transaction amount for each stock I;
    [NET] AMT(I) = START(I) + BUY(I) - SELL(I);
    BOS(I) = BUY(I) + SELL(I); ! Amount bought or sold;
    );

! Overall budget constraint: Ending amount + transaction costs = Sources of
funds;
[BUD] @SUM( ASSET(I): AMT(I) + C(I)*BOS(I) + M(I)*BOS(I)^POW)
    = @SUM( ASSET(i): START(I));
! Expected return target;
```

```
[RET] @SUM( ASSET(J): AMT(J) * RETURN(J)) >= TARGET*@SUM( ASSET(J):
START(J));
END
```

The interesting part of the solution is:

```
  Global optimal solution found.
  Objective value:          0.022633

             Variable           Value
            AMT( ATT)        0.525970
            AMT( GMC)        0.350000
            AMT( USX)        0.123436   .999406
          AMT( TBILL)        0.000000
            BUY( ATT)        0.025970
            BUY( GMC)        0.000000
            BUY( USX)        0.000000
          BUY( TBILL)        0.000000
           SELL( ATT)        0.000000
           SELL( GMC)        0.000000
           SELL( USX)        0.026564
         SELL( TBILL)        0.000000
```

Notice the effect of the market impact term. We bought slightly less of ATT (0.02597 vs. 0.026475), and sold a little less of USX (0.026564 vs. 0.027010). In the model we treated buying and selling in the same way. This seems reasonable, but it is not required. The nonlinear transactions cost model provides some guidance with regard to either unloading a large quantity of a stock, or accumulating a large quantity of a stock.

13.4.3 Portfolios with Taxes

Taxes are an unpleasant complication of investment analysis that should be considered. The effect of taxes on a portfolio is illustrated by the following results during one year for two similar "growth-and-income" portfolios from the Vanguard company. Portfolio S was managed without (Sans) regard to taxes. Portfolio T was managed with after-tax performance in mind:

	Distributions		Initial	
Portfolio	**Income**	**Gain-from-sales**	**Share-price**	**Return**
S	\$0.41	\$2.31	\$19.85	33.65%
T	\$0.28	\$0.00	\$13.44	34.68%

The tax managed portfolio, probably just by chance, in fact had a higher before tax return. It looks even more attractive after taxes. If the tax rate for both dividend income and capital gains is 30%, then the tax paid at year end per dollar invested in portfolio S is $.3 \times (.41 + 2.31) / 19.85 = 4.1$ cents; whereas, the tax per dollar invested in portfolio S is $.3 \times .28/13.44 = 0.6$ of a cent.

Below is a generalization of the Markowitz model to take into account taxes. As input, it requires in particular:

a) number of shares held of each kind of asset,
b) price per share paid for each asset held, and
c) estimated dividends per share for each kind of asset.

The results from this model will differ from a model that does not consider taxes in that this model, when considering equally attractive assets, will tend to:

i. purchase the asset that does not pay dividends, so as to avoid the immediate tax on dividends,
ii. sell the asset that pays dividends, and
iii. sell the asset whose purchase cost was higher, so as to avoid more tax on capital gains.

This is all given that two assets are otherwise identical (presuming rates of return are computed including dividends). For completeness, this model also includes transaction costs and illustrates how a correlation matrix can be used instead of a covariance matrix to describe how assets move together:

```
MODEL:
! Generic Markowitz portfolio model that takes into account
  bid/ask spread and taxes.  (PORTAX)
  Keywords: Markowitz, portfolio, taxes, transaction costs;
 SETS:
  ASSET: RET, START, BUY, SEL, APRICE, BUYAT, SELAT, DVPS, STD, X;
 ENDSETS
 DATA:
! Data based on original Markowitz example;
 ASSET =  TBILL   ATT        GMC      USX;
! The expected returns as growth factors;
   RET  =  1.05  1.089083 1.21367  1.23458;
! S. D. in return for each asset;
   STD =   0     .103959  .241644  .306964;
! Starting composition of the portfolio in shares;
   START =  10     50        70       350;
! Price per share at acquisition;
   APRICE = 1000   80        89        21;
! Current bid/ask price per share;
   BUYAT  = 1000   87        89        27;
   SELAT  = 1000   86        88        26;
! Dividends per share(estimated);
   DVPS =   0     .5        0         0;
! Tax rate;
   TAXR = .32;
! The desired growth factor;
   TARGET = 1.15;
 ENDDATA
```

```
 SETS:
  TMAT( ASSET, ASSET) | &1 #GE# &2: CORR;
 ENDSETS
 DATA:
! Correlation matrix;
   CORR= 1.0
         0  1.000000
         0  0.4938961  1.000000
         0  0.4097276  0.7472293 1.000000 ;
 ENDDATA
!-----------------------------------------------------------------;
!  Min the var in portfolio return;
 [OBJ] MIN =
      @SUM( ASSET( I):  ( X( I)*SELAT( I)* STD( I))^2) +
        2 * @SUM( TMAT( I, J)  | I #NE# J:
           CORR( I, J) * X( I)* SELAT( I) * STD( I)
                      * X( J)* SELAT( J) * STD( J)) ;
! Budget constraint, sales must cover purchases + taxes;
 [BUDC] @SUM( ASSET( I):
          SELAT( I) * SEL( I) - BUYAT( I) * BUY( I)) >= TAXES;
 [TAXC]  TAXES >= TAXR * @SUM( ASSET( I):
           DVPS( I)* X( I) + SEL( I) * ( SELAT( I) - APRICE( I)));
! After tax return requirement.  This assumes we do not pay
   tax on appreciation until we sell;
 [RETC] @SUM( ASSET( I):
         RET( I)* X(I)* SELAT( I)) - TAXES >=
         TARGET * @SUM( ASSET(I): START( I) * SELAT( I));
! Inventory balance for each asset;
  @FOR( ASSET( I):
    [BAL] X( I) = START( I) + BUY( I) - SEL( I);  );
 END
```

13.4.4 Factors Model for Simplifying the Covariance Structure

Sharpe (1963) introduced a substantial simplification to the modeling of the random behavior of stock market prices. He proposed that there is a "market factor" that has a significant effect on the movement of a stock. The market factor might be the Dow-Jones Industrial average, the S&P 500 average, or the Nikkei index. If we define:

M = the market factor,
m_0 = $E(M)$,
s_0^2 = var(M),
e_i = random movement specific to stock i,
s_i^2 = var(e_i).

Sharpe's approximation assumes (where $E(\ )$ denotes expected value):

$E(e_i) = 0$
$E(e_i\, e_j) = 0 \quad$ for $i \neq j$,
$E(e_i\, M) = 0$.

Then, according to the Sharpe single factor model, the return of one dollar invested in stock or asset i is:

$$u_i + b_i M + e_i.$$

The parameters u_i and b_i are obtained by regression (e.g., least squares, of the return of asset i on the market factor). The parameter b_i is known as the "beta" of the asset. Let:

X_i = amount invested in asset i and

define the variance in return of the portfolio as:

$$\begin{aligned} &\text{var}[\textstyle\sum X_i(u_i + b_i M + e_i)] \\ &= \text{var}(\textstyle\sum X_i b_i M) + \text{var}(\textstyle\sum X_i e_i) \\ &= (\textstyle\sum X_i b_i)^2 s_o^2 + \textstyle\sum X_i^2 s_i^2. \end{aligned}$$

Thus, our problem can be written:

$$\begin{aligned} &\text{Minimize} \quad Z^2 s_o^2 + \textstyle\sum X_i^2 s_i^2 \\ &\text{subject to} \\ &\quad Z - \textstyle\sum X_i b_i = 0 \\ &\quad \textstyle\sum X_i = 1 \\ &\quad \textstyle\sum X_i (u_i + b_i m_o) \geq r. \end{aligned}$$

So, at the expense of adding one constraint and one variable, we have reduced a dense covariance matrix to a diagonal covariance matrix.

In practice, perhaps a half dozen factors might be used to represent the "systematic risk". That is, the return of an asset is assumed to be correlated with a number of indices or factors. Typical factors might be a market index such as the S&P 500, interest rates, inflation, defense spending, energy prices, gross national product, correlation with the business cycle, various industry indices, etc. For example, bond prices are very affected by interest rate movements.

13.4.5 Example of the Factor Model

The Factor Model represents the variance in return of an asset as the sum of the variances due to the asset's movement with one or more factors, plus a factor-independent variance.

To illustrate the factor model, we used multiple regression to regress the returns of *ATT*, *GMC*, and *USX* on the S&P 500 index for the same period. The model with solution is:

```
MODEL:
! Multi factor portfolio model;
 SETS:
  ASSET: ALPHA, SIGMA, X;
  FACTOR: RETF, SIGFAC, Z;
  AXF( ASSET, FACTOR): BETA;
 ENDSETS
 DATA:
! The factor(s);
  FACTOR = SP500;
! Mean and s.d. of factor(s);
   RETF = 1.191460;
   SIGFAC = .1623019;
! The stocks were multi-regressed on the factors;
! i.e.:  Return(i) = Alpha(i) + Beta(i) * SP500 + error(i);
   ASSET =    ATT         GMC       USX;
   ALPHA = .563976   -.263502  -.580959;
   BETA  = .4407264  1.23980   1.52384;
   SIGMA = .075817    .125070   .173930;
! The desired return;
   TARGET = 1.15;
 ENDDATA
!----------------------------------------------------;
!  Min the var in portfolio return;
  [OBJ] MIN
      = @SUM( FACTOR( J):( SIGFAC( J) * Z( J))^2)
      + @SUM( ASSET( I): ( SIGMA( I) * X( I))^2) ;
! Compute portfolio betas;
  @FOR( FACTOR( J):
    Z( J) = @SUM( ASSET( I): BETA( I, J) * X( I));
      );
! Budget constraint;
   @SUM( ASSET: X) = 1;
! Return requirement;
    @SUM( ASSET(  I): X( I )* ALPHA( I))
  + @SUM( FACTOR( J): Z( J) * RETF( J)) >= TARGET;
 END
```

Part of the solution is:

```
        Variable           Value      Reduced Cost
          TARGET        1.150000         0.0000000
         X( ATT)       0.5276550         0.0000000
         X( GMC)       0.3736851         0.0000000
         X( USX)       0.9865990E-01     0.0000000
       Z( SP500)       0.8461882         0.0000000
             Row    Slack or Surplus      Dual Price
             OBJ       0.0229409         1.000000
               2       0.0000000         0.3498846
               3       0.0000000         0.3348567
               4       0.0000000        -0.3310770
```

Notice the portfolio makeup is slightly different. However, the estimated variance of the portfolio is very close to our original portfolio.

13.4.6 Scenario Model for Representing Uncertainty

The scenario approach to modeling uncertainty assumes the possible future situations can be represented by a small number of "scenarios". The smallest number used is typically three (e.g., "optimistic," "most likely," and "pessimistic"). Some of the original ideas underlying the scenario approach come from the approach known as stochastic programming; see Madansky (1962), for example. For a discussion of the scenario approach for large portfolios, see Markowitz and Perold (1981) and Perold (1984). For a thorough discussion of the general approach of stochastic programming, see Infanger (1992). Eppen, Martin, and Schrage (1988) use the scenario approach for capacity planning in the automobile industry.

Let:

P_s = Probability scenario s occurs,
u_{is} = return of asset i if the scenario is s,
X_i = investment in asset i,
Y_s = deviation of actual return from the mean if the scenario is s;
$= \sum_i X_i(u_{is} - \sum_q P_q u_{iq})$.

Our problem in algebraic form is:

Minimize $\sum_s P_s Y_s^2$
subject to
$Y_s - \sum_i X_i(u_{is} - \sum_q P_q u_{iq}) = 0$ (deviation from mean of each scenario, s)
$\sum_i X_i = 1$ (budget constraint)
$\sum_i X_i \sum_s P_s u_{is} \geq r$ (desired return).

If asset i has an inherent variability v_i^2, the objective generalizes to:

Min $\sum_i X_i^2 v_i^2 + \sum_s P_s Y_s^2$

The key feature is that, even though this formulation has a few more constraints, the covariance matrix is diagonal and, thus, very sparse.

You will generally also want to put upper limits on what fraction of the portfolio is invested in each asset. Otherwise, if there are no upper bounds or inherent variabilities specified, the optimization will tend to invest in only as many assets as there are scenarios.

13.4.7 Example: Scenario Model for Representing Uncertainty

We will use the original data from Markowitz once again. We simply treat each of the 12 years as being a separate scenario, independent of the other 11 years. Because of the amount of data involved, it is convenient to use the 'sets' form of LINGO in the following model:

```
MODEL:
  ! Scenario portfolio model;
  SETS:
    SCENE: PRB, R, DVU, DVL;
    ASSET:  X;
    SXI( SCENE, ASSET): VE;
  ENDSETS
```

```
  DATA:
   TARGET = 1.15;
    SCENE = 1..12;
   ASSET = ATT GMT USX;
  ! Data based on original Markowitz example;
   VE =
     1.300    1.225    1.149
     1.103    1.290    1.260
     1.216    1.216    1.419
     0.954    0.728    0.922
     0.929    1.144    1.169
     1.056    1.107    0.965
     1.038    1.321    1.133
     1.089    1.305    1.732
     1.090    1.195    1.021
     1.083    1.390    1.131
     1.035    0.928    1.006
     1.176    1.715    1.908;
  ! All scenarios considered to be equally likely;
   PRB= .08333 .08333 .08333 .08333 .08333 .08333
        .08333 .08333 .08333 .08333 .08333 .08333;
  ENDDATA
  ! Target ending value;
   [RET] AVG >= TARGET;
  ! Compute expected value of ending position;
      AVG = @SUM( SCENE: PRB * R);
      @FOR( SCENE( S):
  ! Measure deviations from average;
        DVU( S) - DVL( S) = R(S) - AVG;
  ! Compute value under each scenario;
        R( S) = @SUM( ASSET( J): VE( S, J) * X( J)));
  ! Budget;
   [BUD] @SUM( ASSET: X) = 1;
   [VARI] VAR = @SUM( SCENE: PRB * ( DVU + DVL)^2);
   [SEMIVARI] SEMIVAR = @SUM( SCENE: PRB * (DVL) ^2);
   [DOWNRISK] DNRISK = @SUM( SCENE: PRB * DVL);
  ! Set objective to VAR, SEMIVAR, or DNRISK;
   [OBJ] MIN = VAR;
END
```

When solved, (part of) the solution is:

```
Optimal solution found at step:         4
Objective value:                0.2056007E-01
Variable            Value        Reduced Cost
 X( ATT)        0.5297389           0.0000000
 X( GMT)        0.3566688           0.0000000
 X( USX)        0.1135923           0.0000000
     Row    Slack or Surplus       Dual Price
     RET        0.0000000          -0.3246202
     BUD        0.0000000           0.3321931
     OBJ        0.2056007E-01        1.000000
```

The solution should be familiar. The alert reader may have noticed the solution suggests the same portfolio (except for round-off error) as our original model based on the covariance matrix (based on the same 12 years of data as in the above scenario model). This, in fact, is a general result. In other words, if the covariance matrix and expected returns are calculated directly from the original data by the traditional statistical formulae, then the covariance model and the scenario model, based on the same data, will recommend exactly the same portfolio.

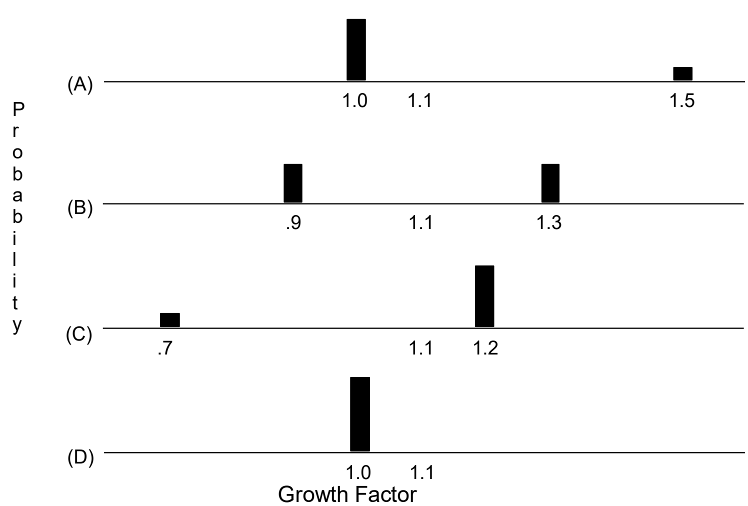
The careful reader will have noticed the objective function from the scenario model (0.02056) is slightly less than that of the covariance model (.02241). The exceptionally perceptive reader may have noticed that except for round-off error, $12 \times 0.02054597/11 = 0.002241$. The difference in objective value is a result simply of the fact that standard statistics packages tend to divide by $N-1$ rather than N when computing variances and covariances, where N is the number of observations. Thus, a slightly more general statement is, if the covariance matrix is computed using a divisor of N rather than $N-1$, then the covariance model and the scenario model will give the same solution, including objective value.

13.5 Measures of Risk other than Variance

The most common measure of risk is variance (or its square root, the standard deviation). This is a reasonable measure of risk for assets that have a symmetric distribution and are traded in a so-called "efficient" market. If these two features do not hold, however, variance has some drawbacks. Consider the four possible growth distributions in Figure 13.2.

Investments *A*, *B*, and *C* are equivalent according to the variance measure because each has an expected growth of 1.10 (an expected return of 10%) and a variance of 0.04 (standard deviation around the mean of 0.20). Risk-averse investors would, however, probably not be indifferent among the three. Under distribution (*A*), you would never lose any of your original investment, and there is a 0.2 probability of the investment growing by a factor of 1.5 (i.e., a 50% return). Distribution (*C*), on the other hand, has a 0.2 probability of an investment decreasing to 0.7 of its original value (i.e., a negative 30% return). Risk-averse investors would tend to prefer (*A*) most and to prefer (*C*) least. This illustrates variance need not be a good measure of risk if the distribution of returns is not symmetric:

Figure 13.2 Possible Growth Factor Distributions



Investment (*D*) is an inefficient investment. It is dominated by (*A*). Suppose the only investments available are (*A*) and (*D*) and our goal is to have an expected return of at least 5% (i.e., a growth factor of 1.05) and the lowest possible variance. The solution is to put 50% of our investment in each of (*A*) and (*D*). The resulting variance is 0.01 (standard deviation = 0.1). If we invested 100% in (*A*), the standard deviation would be 0.20. Nevertheless, we would prefer to invest 100% in (*A*). It is true the return is more random. However, our profits are always at least as high under every outcome. (If the randomness in profits is an issue, we can always give profits to a worthy educational institution when our profits are high to reduce the variance.) Thus, the variance objective may cause us to choose inefficient investments.

In active and efficient markets such as major stock markets, you will tend not to find investments such as (*D*) because investors will realize (*A*) dominates (*D*). Thus, the market price of (*D*) will drop until its return approaches competing investments. In investment decisions regarding new physical facilities, however, there are no strong market forces making all investment candidates "efficient", so the variance risk measure may be less appropriate in such situations.

13.5.1 Value at Risk(VaR)

In 1994, J.P. Morgan popularized the "Value at Risk" (VaR) concept with the introduction of their RiskMetrics™ system. To use VaR, you must specify two numbers: 1) an interval of time, typically one day or ten days, over which you are concerned about losing money, and 2) a probability threshold, typically 5% (or 1%), beyond which you care about harmful outcomes. VaR is then defined as that amount of loss in one day that has at most a 5% (or 1%) probability of being exceeded. A comprehensive survey of VaR is Jorion (2001). Some of the popularity of VaR results from the fact that it is a method recommended as part of the Basel Accord for measuring the risk of the portfolios of European banks. Banks must hold capital reserves proportional to their risk, e.g., as measured by VaR.

Example

Suppose that one day from now we think that our portfolio will have appreciated in value by $12,000. The actual value, however, has a Normal distribution with a standard deviation of $10,000. From a Normal table, we can determine that a left tail probability of 5% corresponds to an outcome that is 1.644853 standard deviations below the mean. Now:

$$12000 - 1.644853 * 10000 = -4448.50.$$

So, we would say that the value at risk is $4448.50.

13.5.2 Example of VaR

Let us apply the VaR approach to our standard example, the ATT/GMC/USC model. Suppose that our risk tolerance is 5% and we want to minimize the value at risk of our portfolio. This is equivalent to maximizing that threshold, so the probability our wealth is below this threshold is at most .05.

Analysis:

If the end-of-year portfolio value has a Normal distribution, then a left tail probability of 5% corresponds to a point that is 1.64485 standard deviations below the mean. Minimizing the value at risk corresponds to choosing the mean and standard deviation of the portfolio, so the (mean – 1.64485 * (standard deviation)) is maximized. The following model will do this:

```
MODEL: ! Markowitz Value at Risk Portfolio Model(PORTVAR);
 SETS:
  STOCKS:  X, RET;
  COVMAT(STOCKS, STOCKS): VARIANCE;
 ENDSETS
 DATA:
  STOCKS   =    ATT        GMC      USX;
!Covariance matrix and expected returns;
  VARIANCE = .01080754 .01240721 .01307513
             .01240721 .05839170 .05542639
             .01307513 .05542639 .09422681 ;
       RET = 1.0890833  1.213667  1.234583 ;
    STARTW = 1.0;  ! How much we start with;
       RHO = .05;! Risk tolerance, must be < .5;
 ENDDATA
!----------------------------------------------------------;
! Get the s.d. corresponding to this risk threshold;
    RHO = @PSN( Z);
  @FREE( Z);
! Maximize value not at risk;
[VAR] Max = ARET + Z * SD;
  ARET = @SUM( STOCKS:  X * RET) ;
! The variance ( or SD^2) of the portfolio must be this large;
  SD^2 >= @SUM( COVMAT(I, J):  X(I) *  X(J) * VARIANCE(I, J));
!  Use exactly 100% of the starting budget;
[BUDGET] @SUM( STOCKS:  X) = STARTW;
END
```

With solution:

```
Global optimal solution found.
Objective value:                            0.9257590
Elapsed runtime seconds:                         0.16
Model is a second-order cone
```

```
Variable            Value          Reduced Cost
     RHO         0.050000          0.0000000
       Z        -1.644853          0.0000000
    ARET         1.109300          0.0000000
      SD         0.111585          0.0000000
X( ATT)         0.843034          0.0000000
X( GMC)         0.125330          0.0000000
X( USX)         0.031636          0.0000000
RET( ATT)        1.089083          0.0000000
RET( GMC)        1.213667          0.0000000
RET( USX)        1.234583          0.0000000

     Row     Slack or Surplus        Dual Price
       1       -0.4163336E-16         -1.081707
     VAR        0.9257590              1.000000
       3       -0.2220446E-15          1.000000
       4        0.0000000             -1.644853
  BUDGET        0.0000000              0.9257590
```

Note that, if we invested solely in ATT, the portfolio variance would be .01080754. So, the standard deviation would be .103959, and the VAR would be 1 - (1.089083 - 1.644853 * .103959) = .0818.

The portfolio is efficient because it is maximizing a weighted combination of the expected return and (a negatively weighted) standard deviation. Thus, if there is a portfolio that has both higher expected return and lower standard deviation, then the above solution would not maximize the objective function above.

Note, if you use a risk tolerance: *RHO* = .1988, then you get essentially the original portfolio considered for the ATT/GMC/USX problem.

There are two things to note about the heading of the solution report: 1) The solution is labelled with the heading "Global optimal solution found" and 2) the model type is described as "second-order cone. The constraint

```
SD^2 >= @SUM( COVMAT(I, J):  X(I) *  X(J) * VARIANCE(I, J));
```

is a form of what is called a second-order cone constraint, or SOC for short. LINGO is able to identify such constraints, and if all the constraints are either linear or second-order cone constraints, then LINGO can solve large problems of that type fast and solve them to a global, not just local optimum.

13.5.3 VaR Anomalies

If you want just a single number to describe risk, VaR is a useful, easy to understand metric. You should not, however, use VaR without considering its anomalous features. The most obvious criticism of VaR is that it gives attention to only one percentile point of the portfolio return distribution. It does not pay attention to how really bad a low probability event might be. Two portfolios P1 and P2 may each have a probability of at most 5% of losing $1M or more, so the VaR is the same for them. Suppose, however, that P1 has a probability of 5% of losing exactly $1M and no more, whereas P2 has a probability of 1% of losing exactly $1M and a probability of 4% of losing $10M. Most people would consider P2 as the riskier one. This "narrow-mindedness" of VaR leads to several questionable features: a) [Good News anomaly] If we change a parameter of a candidate investment for our portfolio so that the investment now pays off more, then a VaR objective may suggest that we invest less in that investment after the change; b) [Diversification is Bad anomaly] If bank 1, with portfolio X1 takes over bank 2 with its portfolio X2, then we may find that VaR(X1 + X2) > VaR(X1) + VaR(X2), i.e., diversification may appear to increase risk according to the VaR measure.

We first illustrate anomally (a) above. A very conservative investor might react to risk by maximizing the minimum return over scenarios. This is equivalent to the VaR approach in which we set the risk tolerance to arbitrarily close to 0. There are some curious implications from this. Suppose the only investments available are *A* and *C* above and the two scenarios are:

Scenario	Probability	Payoff from A	Payoff from C
1	0.8	1.0	1.2
2	0.2	1.5	0.7

If we wish to maximize the minimum possible wealth, the probability of a scenario does not matter, as long as the probability is positive. Thus, the following LP is appropriate:

```
 MAX = WMIN;
!  Initial budget constraint;
              A +       C = 1;
!  Wealth under scenario 1;
- WMIN +       A + 1.2 * C >= 0;
!  Wealth under scenario 2;
- WMIN + 1.5 * A + 0.7 * C >= 0;
```

The solution is:

```
Objective value:                  1.100000
Variable           Value        Reduced Cost
    WMIN        1.100000           0.0000000
       A       0.5000000           0.0000000
       C       0.5000000           0.0000000
```

Given that both investments have an expected return of 10%, it is not surprising the expected growth factor is 1.10. That is, a return of 10%. The possibly surprising thing is there is no risk. Regardless of which scenario occurs, the $1 initial investment will grow to $1.10 if 50 cents is placed in each of *A* and *C*.

Now, suppose an extremely reliable friend provides us with the interesting news that, “if scenario 1 occurs, then investment *C* will payoff 1.3 rather than 1.2”. This is certainly good news. The expected return for *C* has just gone up, and its downside risk has certainly not gotten worse. How should we react to it? We make the obvious modification in our model:

```
MAX = WMIN;
!  Initial budget constraint;
               A +       C = 1;
!  Wealth under scenario 1;
- WMIN +       A + 1.3 * C > 0;
!  Wealth under scenario 2;
- WMIN + 1.5 * A + 0.7 * C > 0;
```

and re-solve it to find:

```
Objective value:                  1.136364
Variable           Value        Reduced Cost
    WMIN        1.136364           0.0000000
       A       0.5454545           0.0000000
       C       0.4545455           0.0000000
```

This is a bit curious. We have decreased our investment in *C*. This is as if our friend had continued on: “I have this very favorable news regarding stock *C*. Let’s sell it before the market has a chance to react”. Why the anomaly? The problem is we are basing our measure of goodness on a single point among the possible payoffs. In this case, it is the worst possible. For a further discussion of these issues, see Clyman (1995).

Now let’s illustrate feature (b), the “Diversification is Bad anomaly”. Suppose that both portfolios X1 and X2 have a beginning wealth of 100 and have independent, identically distributed distributions of ending wealth, w, of Prob$\{w = 80\}$ = .04, and Prob$\{w = 110\}$ = .96. Thus, at a risk tolerance of 5%, both portfolios have a VaR = 0, i.e., the probability of losing 0 money or more is less than or equal to 5%. If the two portfolios are combined, the beginning value is 200, and the possible ending values and probabilities are Prob$\{w = 160\}$ = .0016; Prob$\{w = 190\}$ = .0768; and Prob$\{w = 220\}$ = .9216. Now the VaR at the 5% level is 200 – 190 = 10. The VaR of the merged bank is greater than the sum of the VaRs of the individual banks. The amount of safety capital the two banks would have to carry would be greater in total after the merger according to VaR rules.

13.5.4 Conditional Value at Risk(CVaR)

We saw that a weakness of VaR is that it does not pay attention to how bad a low probability event can be. CVaR, see Palmquist, Uryasev, and Krokhmal(2002), corrects this deficiency. Once again, suppose portfolio P1 has a probability of 5% of losing exactly \$1M and no more, whereas P2 has a probability of 1% of losing exactly \$1M and a probability of 4% of losing \$10M. According to VaR, we would be indifferent between P1 and P2 because at the 5% risk tolerance, they both have a VaR of \$1M. Conditional Value at Risk(CVaR) explicitly takes into account the amount by which the loss exceeds the VaR threshold. Similar to VaR, CVaR requires us to specify a risk tolerance ρ, e.g., 5%. Optionally, we may specify an expected return preference $\alpha \geq 0$. If the random variable w is the final wealth of the portfolio, then CVaR chooses a portfolio and VaR threshold, t, so as to maximize a weighted combination of: the final portfolio value, the VaR value, and minus the expected amount by which the final portfolio falls short of the VaR target. Algebraically, the CVaR objective is:

$$\text{Max } \alpha \, \mathrm{E}(w) + \rho \, t - \mathrm{E}(\max[0, t - w]).$$

The variable t should not appear in any other constraints. It is fairly easy to show that at an optimum, t will have the feature that $\text{Prob}\{w < t\} \leq \rho \leq \text{Prob}\{w \leq t\}$. That is, for the optimal portfolio, its VaR will be: initial wealth – t. The following model illustrates the determination of a CVaR portfolio:

```
MODEL:
! Scenario portfolio model;
! Minimize the Conditional Value at Risk;
 SETS:
  SCENE: PRB, W, DVU, DVL;
  ASSET:  X;
  SXI( SCENE, ASSET): VE;
 ENDSETS
DATA:
 RHO = .1; ! Risk tolerance;
 ALPHA = 0;
 TARGET = 1.15;
 SCENE = 1..12;
 ASSET =
    ATT     GMC       USX;
! Data based on original Markowitz example;
VE =
   1.300    1.225    1.149
   1.103    1.290    1.260
   1.216    1.216    1.419
   0.954    0.728    0.922
   0.929    1.144    1.169
   1.056    1.107    0.965
   1.038    1.321    1.133
   1.089    1.305    1.732
   1.090    1.195    1.021
   1.083    1.390    1.131
   1.035    0.928    1.006
   1.176    1.715    1.908;
! All scenarios happen to be equally likely;
 PRB= .0833333 .0833333 .0833333 .0833333 .0833333
      .0833333 .0833333 .0833333 .0833333 .0833333
      .0833333 .0833333;
ENDDATA
! Compute portfolio value under each scenario;
  @FOR(SCENE(S):W(S) = @SUM(ASSET(J):VE(S,J) * X(J));
! Measure deviations from CVaR target T;
     DVL( S) - DVU( S) = T - W(S) ;
      );
! Budget;
 [BUD] @SUM( ASSET(i): X(i)) = 1;
! Compute expected value of ending position;
 [DEFAVG] AVG = @SUM( SCENE(s): PRB(s) * W(s));
! Ending value >= target ;
 [RET] AVG >= TARGET;
! Minimize conditional value at risk;
 [OBJ] MAX = ALPHA*AVG + RHO*T - @SUM( SCENE(s): PRB(s)* DVL(s));
END
```

Part of the solution is:

```
 Objective value:    0.09534855

             Variable          Value
                  RHO      0.1000000
                ALPHA       0.000000
               TARGET       1.150000
                    T       1.017901
                  AVG       1.150000
                W( 1)       1.236780
                W( 2)       1.168732
                W( 3)       1.300991
                W( 4)       0.940602
                W( 5)       1.029482
                W( 6)       1.017901
                W( 7)       1.077774
                W( 8)       1.358208
                W( 9)       1.061111
               W( 10)       1.103096
               W( 11)       1.022858
               W( 12)       1.482470
              X( ATT)       0.581326
              X( GMC)       0.000000
              X( USX)       0.418674
```

The initial value of this portfolio was 1, so the VaR of this portfolio is 1 – T = -.017901. There are 12 scenarios. Notice that in only 1 of the 12, scenario 4, is the final wealth less than T = 1.017901. Thus, in 1 outcome out of 12, or less than 10% of the outcomes, would the final value be less than 1.017901.

13.6 Scenario Model and Minimizing Downside Risk

Minimizing the variance in return is appropriate if either:

1) the actual return is Normal-distributed or
2) the portfolio owner has a quadratic utility function.

In practice, it is difficult to show either condition holds. Thus, it may be of interest to use a more intuitive measure of risk. One such measure is the downside risk, which intuitively is the expected amount by which the return is less than a specified target return. The approach can be described if we define:

T = user specified target threshold. When risk is disregarded, this is typically less than the maximum expected return and greater than the return under the worst scenario.

Y_s = amount by which the return under scenario s falls short of target.

$= \max\{0, T - \sum X_i u_{is}\}$

The model in algebraic form is then:

$$\text{Min} \sum P_s \, Y_s \qquad \text{! Minimize expected downside risk}$$

subject to

(compute deviation below target of each scenario, s):

$$Y_s - T + \sum X_i \, u_{is} \geq 0$$

$$\sum X_i = 1 \qquad \text{(budget constraint)}$$

$$\sum X_\text{i} \sum P_s \, u_{is} \geq r \qquad \text{(desired return).}$$

Notice this is just a linear program.

13.6.1 Semi-variance and Downside Risk

The most common alternative suggested to variance as a measure of risk is some form of downside risk. One such measure is semi-variance. It is essentially variance, except only deviations below the mean are counted as risk. The scenario model is well suited to such measures. The previous scenario model needs only a slight modification to convert it to a semi-variance model. The Y variables are redefined to measure the deviation below the mean only, zero otherwise. The resulting model is:

```
MODEL:
  ! Scenario portfolio model;
  ! Minimize the semi-variance;
   SETS:
    SCENE/1..12/: PRB, R, DVU, DVL;
    ASSET/ ATT,   GMT,   USX/:  X;
    SXI( SCENE, ASSET): VE;
   ENDSETS
  DATA:
   TARGET = 1.15;
  ! Data based on original Markowitz example;
  VE =
     1.300    1.225    1.149
     1.103    1.290    1.260
     1.216    1.216    1.419
     0.954    0.728    0.922
     0.929    1.144    1.169
     1.056    1.107    0.965
     1.038    1.321    1.133
     1.089    1.305    1.732
     1.090    1.195    1.021
     1.083    1.390    1.131
     1.035    0.928    1.006
     1.176    1.715    1.908;
  ! All scenarios happen to be equally likely;
   PRB= .0833333 .0833333 .0833333 .0833333 .0833333
        .0833333 .0833333 .0833333 .0833333 .0833333
        .0833333 .0833333;
  ENDDATA
  ! Compute value under each scenario;
    @FOR(SCENE(S):R(S)  = @SUM(ASSET(J):VE(S,J)  * X(J));
  ! Measure deviations from average;
        DVU( S) - DVL( S) = R(S) - AVG;);
```

```
  ! Budget;
   [BUD] @SUM( ASSET: X) = 1;
  ! Compute expected value of ending position;
   [DEFAVG] AVG = @SUM( SCENE: PRB * R);
  ! Target ending value;
   [RET] AVG > TARGET;
  ! Minimize the  semi-variance;
   [OBJ] MIN = @SUM( SCENE: PRB * DVL^2);
END
```

The resulting solution is:

```
Optimal solution found at step:         4
Objective value:                0.8917110E-02
Variable          Value          Reduced Cost
   R( 1)       1.238875           0.0000000
   R( 2)       1.170760           0.0000000
   R( 3)       1.294285           0.0000000
   R( 4)      0.9329399           0.0000000
   R( 5)       1.029848           0.0000000
   R( 6)       1.022875           0.0000000
   R( 7)       1.085554           0.0000000
   R( 8)       1.345299           0.0000000
   R( 9)       1.067442           0.0000000
  R( 10)       1.113355           0.0000000
  R( 11)       1.019688           0.0000000
  R( 12)       1.479083           0.0000000
 DVU( 1)      0.8887491E-01       0.0000000
 DVU( 2)      0.2076016E-01       0.0000000
 DVU( 3)      0.1442846           0.0000000
 DVU( 4)      0.0000000           0.3617666E-01
 DVU( 5)      0.0000000           0.2002525E-01
 DVU( 6)      0.0000000           0.2118756E-01
 DVU( 7)      0.0000000           0.1074092E-01
 DVU( 8)      0.1952993           0.0000000
 DVU( 9)      0.0000000           0.1375965E-01
DVU( 10)      0.0000000           0.6107114E-02
DVU( 11)      0.0000000           0.2171863E-01
DVU( 12)      0.3290833           0.0000000
 DVL( 1)      0.0000000           0.8673617E-09
 DVL( 2)      0.0000000           0.8673617E-09
 DVL( 3)      0.0000000           0.8673617E-09
 DVL( 4)      0.2170601           0.0000000
 DVL( 5)      0.1201515           0.0000000
 X( ATT)      0.5757791           0.0000000
 X( GMT)      0.3858243E-01       0.0000000
 X( USX)      0.3856385           0.0000000
     Row    Slack or Surplus      Dual Price
     BUD      0.0000000           0.1198420
  DEFAVG      0.0000000         -0.9997334E-02
     RET      0.0000000         -0.1197184
     OBJ      0.8917110E-02        1.000000
```

Notice the objective value is less than half that of the variance model. We would expect it to be at most half, because it considers only the down (not the up) deviations. The most noticeable change in the portfolio is substantial funds have been moved to *USX* from *GMC*. This is not surprising if you look at the original data. In the years in which *ATT* performs poorly, *USX* tends to perform better than *GMC*.

13.6.2 Downside Risk and MAD

If the threshold for determining downside risk is the mean return, then minimizing the downside risk is equivalent to minimizing the mean absolute deviation (MAD) about the mean. This follows easily because the sum of deviations (not absolute) about the mean must be zero. Thus, the sum of deviations above the mean equals the sum of deviations below the mean. Therefore, the sum of absolute deviations is always twice the sum of the deviations below the mean. Thus, minimizing the downside risk below the mean gives exactly the same recommendation as minimizing the sum of absolute deviations below the mean. Konno and Yamazaki (1991) use the MAD measure to construct portfolios from stocks on the Tokyo stock exchange.

13.6.3 Power and Log Utility Functions

Downside risk and semi-variance are examples of utility functions. The basic idea is that the utility of having an extra dollar of wealth depends upon the amount of wealth we already have. Almost all people feel that: a) more wealth is better. Many people are risk averse in the sense that they feel that: b) not losing a dollar is more important than gaining an extra dollar. One of the simplest utility functions that can capture (a) and (b) is the power utility function, wherein the value of having wealth w is essentially proportional to w raised to the power γ. Usually the function is "normalized" by subtracting 1 and dividing by γ, so the utility of having wealth w is:

$U(w) = (w^{\gamma} - 1)/\gamma$;

An investor is said to be risk averse if $0 \leq \gamma < 1$, risk neutral if $\gamma = 1$, and risk preferring if $\gamma > 1$. As γ approaches 0, the power utility function approaches the log utility:

$U(w) = \text{LN}(w)$, where LN is the natural logarithm.

Recalling our little four alternatives from Figure 13.2, the utilities for various γ are shown below.

	γ=1	γ=0.5	γ=0.1	γ=0.01	Log
A)	0.1	0.089898	0.082759	0.081258	0.081093
B)	0.1	0.088859	0.080514	0.078702	0.078502
C)	0.1	0.087376	0.077117	0.074782	0.074522
D)	0	0			

For example, 0.089898 = 0.8*(1^0.5 -1)/0.5 + 0.2*(1.5^0.5-1)/0.5. Notice that when $\gamma < 1$, alternative A is the preferred alternative, which is consistent with what most people would choose. Also notice that as γ approaches 0, the power utility approaches the Log utility.

A utility function such as the power utility provides some advice for the "bet sizing" or "how much of our wealth should we put at risk" problem. Suppose our current wealth is $1000 and we have two alternatives: i) put money in the mattress, or ii) a bet for which for every $1 invested, our investment either grows to $2 with probability 0.6, or we lose our entire investment with probability 0.4. Note that this is essentially equivalent to our offering an insurance policy where the premium is $1 and the amount insured is $2, and the probability we will have to pay out the insured amount is 0.4. The expected net return is 0.6*2 + 0.4*0 – 1 = 0.2. This is positive, so if we are risk neutral we put all of money in the risky alternative. If we lose, however, we are out of business, so we may want to rethink this. The relevant portfolio model if we use a power utility function is:

```
! The bet sizing problem, using Power Utility;
GAMMA = 0.01;   ! We are rather risk averse;
WEALTH = 1000; ! Our current wealth;
PWIN = 0.6;     ! Prob(Win bet);
! Some of our money goes in the Matress,
  some goes to the Bet;
M + B = WEALTH;
! We want to maximize our expected utility, using a Power utility;
MAX =    PWIN*((M+2*B)^GAMMA - 1)/GAMMA ! Bet pays off;
   + (1-PWIN)*((M+0*B)^GAMMA - 1)/GAMMA; !Lose bet;
```

The solution is:

```
Global optimal solution found.
Objective value:       7.173722
 Variable            Value
        M            798.0349
        B            201.9651
```

So we risk about 20% of our wealth on the bet. You can verify that if you increase your risk tolerance to $\gamma = 0.63$, then we would invest about \$500 of our \$1000 in the bet.

13.6.4 Scenarios Based Directly Upon a Covariance Matrix

If only a covariance matrix is available, rather than original data, then, not surprisingly, it is nevertheless possible to construct scenarios that match the covariance matrix. The following example uses just four scenarios to represent the possible returns from the three assets: *ATT*, *GMC*, and *USX*. These scenarios have been constructed, using the methods of section 12.8.2, so they mimic behavior consistent with the original covariance matrix:

```
MODEL:
SETS:
! Each asset has a variable value and an average return;
 ASSET:  X, RET;
! the variance of return at each scenario (which can be negative), and
the probability of it happening;
 SCEN: Y, P;
! Return for each asset under each scenario;
 COVMAT( SCEN, ASSET):ENTRY;
ENDSETS
DATA:
  P = .25 .25 .25 .25; ! Four equi-likely scenarios;
 ASSET =   ATT       GMC       USX;
 ENTRY =0.9851237 1.304437  1.097669
        1.193042  1.543131  1.756196
        0.9851237 0.8842088 1.119948
        1.193042  1.122902  0.9645076;
  RET = 1.089083  1.213667  1.234583;
```

```
ENDDATA
! Minimize the variance;
MIN = @SUM( SCEN(s): Y(s) * Y(s) * P(s));
!  Compute the deviation from mean under each scenario;
@FOR(SCEN(s):Y(s) = @SUM(ASSET(J): ENTRY(s,J)* X(J)) - MEAN
    );
! The Budget constraint;
@SUM(ASSET(j):  X(j)) = 1;
! Define or compute the mean;
@SUM(ASSET(j):  X * RET) = MEAN;
MEAN > 1.15;! Target return;
! The variance of each return can be negative;
@FOR(SCEN: @FREE(Y));
END
```

When solved, we get the familiar solution:

```
Optimal solution found at step:          4
Objective value:                  0.2241380E-01
      Variable           Value      Reduced Cost
          MEAN        1.150000         0.0000000
       X( ATT)       0.5300912         0.0000000
       X( GMC)       0.3564126         0.0000000
       X( USX)       0.1134962         0.0000000
     RET( ATT)        1.089083         0.0000000
     RET( GMC)        1.213667         0.0000000
     RET( USX)        1.234583         0.0000000
         Y( 1)     -0.3829557E-01      0.0000000
         Y( 2)       0.2317340         0.0000000
         Y( 3)      -0.1855416         0.0000000
         Y( 4)     -0.7894565E-02      0.0000000
         P( 1)       0.2500000         0.0000000
         P( 2)       0.2500000         0.0000000
         P( 3)       0.2500000         0.0000000
         P( 4)       0.2500000         0.0000000
ENTRY( 1, ATT)       0.9851237         0.0000000
ENTRY( 1, GMC)        1.304437         0.0000000
ENTRY( 1, USX)        1.097669         0.0000000
ENTRY( 2, ATT)        1.193042         0.0000000
ENTRY( 2, GMC)        1.543131         0.0000000
ENTRY( 2, USX)        1.756196         0.0000000
ENTRY( 3, ATT)       0.9851237         0.0000000
ENTRY( 3, GMC)       0.8842088         0.0000000
ENTRY( 3, USX)        1.119948         0.0000000
ENTRY( 4, ATT)        1.193042         0.0000000
ENTRY( 4, GMC)        1.122902         0.0000000
ENTRY( 4, USX)       0.9645076         0.0000000
           Row    Slack or Surplus    Dual Price
             1       0.2241380E-01     1.000000
             2       0.0000000         0.1914778E-01
             3       0.0000000        -0.1158670
             4       0.0000000         0.9277079E-01
             5       0.0000000         0.3947280E-02
             6       0.0000000         0.3621391
             7       0.0000000        -0.3538852
             8       0.0000000          -0.3538841
```

Notice the objective function value and the allocation of funds over *ATT*, *GMC*, and *USX* are essentially identical to our original portfolio example.

13.7 Hedging, Matching and Program Trading

13.7.1 Portfolio Hedging

Given a "benchmark" portfolio *B*, we say we hedge *B* if we construct another portfolio *C* such that, taken together, *B* and *C* have essentially the same return as *B*, but lower risk than *B*. Typically, our portfolio *B* contains certain components that cannot be removed. Thus, we want to buy some components negatively correlated with the existing ones. Examples are:

a) An airline knows it will have to purchase a lot of fuel in the next three months. It would like to be insulated from unexpected fuel price increases.
b) A farmer is confident his fields will yield $200,000 worth of corn in the next two months. He is happy with the current price for corn. Thus, would like to "lock in" the current price.

13.7.2 Portfolio Matching, Tracking, and Program Trading

Given a benchmark portfolio *B*, we say we construct a matching or tracking portfolio if we construct a new portfolio *C* that has stochastic behavior very similar to ***B***, but excludes certain instruments in *B*. Example situations are:

a) A portfolio manager does not wish to look bad relative to some well-known index of performance such as the S&P 500, but for various reasons cannot purchase certain instruments in the index.
b) An arbitrageur with the ability to make fast, low-cost trades wants to exploit market inefficiencies (i.e., instruments mispriced by the market). If he can construct a portfolio that perfectly matches the future behavior of the well-defined portfolio, but costs less today, then he has an arbitrage profit opportunity (if he can act before this "mispricing" disappears).
c) A retired person is concerned mainly about inflation risk. In this case, a portfolio that tracks inflation is desired.

As an example of (a), a certain so-called "green" mutual fund will not include in its portfolio companies that derive more than 2% of their gross revenues from the sale of military weapons, own directly or operate nuclear power plants, or participate in business related to the nuclear fuel cycle.

The following table, for example, compares the performance of six Vanguard portfolios with the indices the portfolios were designed to track; see Vanguard (1995):

Total Return Six Months Ended June 30, 1995

Vanguard Portfolio Name	Portfolio Growth	Comparative Growth Index	Index Name
500 Portfolio	+20.1%	+20.2%	S&P500
Growth Portfolio	+21.1	+21.2	S&P500/BARRA Growth
Value Portfolio	+19.1	+19.2	S&P500/BARRA Value
Extended Market Portfolio	+17.1%	+16.8%	Wilshire 4500 Index
SmallCap Portfolio	+14.5	+14.4	Russell 2000 Index
Total Stock Market Portfolio	+19.2%	+19.2%	Wilshire 5000 Index

Notice, even though there is substantial difference in the performance of the portfolios, each matches its benchmark index quite well.

13.8 Methods for Constructing Benchmark Portfolios

A variety of approaches has been used for constructing hedging and matching portfolios. For matching portfolios, an intuitive approach has been to generalize the Markowitz model, so the objective is to minimize the variance in the difference in return between the target portfolio and the tracking portfolio.

A useful way to think about hedging or matching of a benchmark is to think of it as our being forced to include the benchmark or its negative in our portfolio. Suppose the benchmark is a simple index such as the S&P500. If our measure of risk is variance, then proceed as follows:

1. Include the benchmark in the covariance matrix just like any other instrument, except do not include it in the budget constraint. We presume we have a budget of $1 to invest in the controllable, non-benchmark portion of our portfolio.
2. To get a "matching" portfolio (e.g., one that mimics the S&P 500), set the value of the benchmark factor to −1. The essential effect is the off diagonal covariance terms are negated in the row/column of the benchmark factor. Effectively, we have shorted the factor. If we can get the total variance to zero, we have perfectly matched the randomness of the benchmark.
3. To get a "hedging" portfolio (e.g., one as negatively correlated with the S&P 500 as possible), set the value of the benchmark factor to +1. Thus, we will compose the rest of the portfolio to counteract the effect of the factor we are stuck with having in the portfolio.

One might even want to drop the budget constraint. The solution will then tell you how much to invest in the controllable portfolio to get the best possible hedge or match per $ of the benchmark.

The following model illustrates the extension of the Markowitz approach to the hedging case where we want to “cancel out” some benchmark. In the case of *GMC*, it could be that our decision maker works for *GMC* and thus has his fortunes unavoidably tied to those of *GMC*. He might wish to find a portfolio negatively correlated with *GMC*:

```
MODEL:
  !Generic Markowitz portfolio Hedging model(PORTHEDG);
  ! We want to hedge the first or "benchmark" asset
    with the remaining ones;
   SETS:
    ASSET/ GMC  ATT  USX/: RET, X;
    TMAT( ASSET, ASSET) | &1 #GE# &2: COV;
   ENDSETS
   DATA:
  ! The expected returns;
     RET = 1.21367,   1.089083,   1.23458;
  ! Covariance matrix;
     COV =
           .05839170
           .01240721  .01080754
           .05542639  .01307513  .09422681;
   ! The desired return;
     TARGET = 1.15;
   ENDDATA
  !-------------------------------------------------;
  !  Min the var in portfolio return;
   [OBJ] MIN= ( @SUM( ASSET( I):
                  COV( I, I) * X( I)^2) +
           2 * @SUM( TMAT( I, J) | I #NE# J:
               COV( I, J) * X( I) * X( J))) ;
  !We are stuck with the first asset in the portfolio;
   X( 1) = 1;
  !  Budget constraint(applies to remaining assets);
   [BUDGET] @SUM( ASSET( I)| I #GT# 1: X( I)) = 1;
  !  Return requirement(applies to remaining assets);
   [RETURN] @SUM( ASSET( I)| I #GT# 1:
                RET( I) * X( I)) >= TARGET;
END
```

The solution is:

```
Optimal solution found at step:         4
Objective value:                0.1457632
Variable           Value        Reduced Cost
 X( GMC)        1.000000           0.0000000
 X( ATT)       0.5813178           0.0000000
 X( USX)       0.4186822           0.0000000
```

Thus, our investor puts more of the portfolio in *ATT* than in *USX* (whose fortunes are more closely tied to those of *GMC*).

The following model illustrates the extension of the Markowitz approach to the matching case where we want to construct a portfolio that *mimics or matches* a benchmark portfolio. In this case, we want to match the S&P500, but limit ourselves to investing in only *ATT*, *GMC*, and *USX*:

```
MODEL:
  !Gen. Markowitz portfolio Matching model(PORTMTCH);
  ! We want to match the first or "benchmark" asset
    with the remaining ones;
   SETS:
    ASSET/ SP500 ATT GMC  USX/: RET, X;
    TMAT( ASSET, ASSET) | &1 #GE# &2: COV;
   ENDSETS
   DATA:
  ! The expected returns;
     RET = 1.191458  1.089083, 1.21367, 1.23458;
  ! Covariance matrix;
     COV =
       .02873661
       .01266498   .01080754
       .03562763   .01240721   .05839170
       .04378880   .01307513   .05542639   .09422681;
   ! The desired return;
     TARGET = 1.191458;
   ENDDATA
  !-------------------------------------------------;
  !  Min the var in portfolio return;
   [OBJ] MIN = (@SUM( ASSET(I): COV(I, I) * X( I)^2)
           + 2 * @SUM( TMAT( I, J) | I #NE# J:
                       COV( I, J) * X( I) * X( J))) ;
  !Matching is equivalent to being short the benchmark;
     X( 1) = -1;
     @FREE( X( 1));
  !  Budget constraint(applies to remaining assets);
   [BUDGET] @SUM( ASSET( I)| I #GT# 1: X( I)) = 1;
  !  Return requirement(applies to remaining assets);
   [RETURN] @SUM( ASSET( I)| I #GT# 1:
                         RET( I) * X( I)) >= TARGET;
END
```

The solution is:

```
Optimal solution found at step:         4
Objective value:                0.5245968E-02
 Variable           Value        Reduced Cost
X( SP500)       -1.000000           0.0000000
  X( ATT)       0.2276635           0.0000000
  X( GMC)       0.4781277           0.0000000
  X( USX)       0.2942089         -0.1266506E-07
```

13.8.1 Scenario Approach to Benchmark Portfolios

If we use the scenario approach, then the hedging model looks as follows:

```
MODEL:      ! (PRTSHDGE);
  ! Scenario portfolio model, Hedge 1st asset;
  ! Minimize the variance;
   SETS:
    SCENE/1..12/: PRB, R, DVU, DVL;
    ASSET/  GMT, ATT,  USX/:  X;
    SXA( SCENE, ASSET): VE;
   ENDSETS
   DATA:
  ! Data based on original Markowitz example;
   VE =
     1.225   1.300    1.149
     1.290   1.103    1.260
     1.216   1.216    1.419
     0.728   0.954    0.922
     1.144   0.929    1.169
     1.107   1.056    0.965
     1.321   1.038    1.133
     1.305   1.089    1.732
     1.195   1.090    1.021
     1.390   1.083    1.131
     0.928   1.035    1.006
     1.715   1.176    1.908;
  ! All scenarios happen to be equally likely;
   PRB= .0833333 .0833333 .0833333 .0833333 .0833333
        .0833333 .0833333 .0833333 .0833333 .0833333
        .0833333 .0833333;
  ! The desired return;
     TARGET = 1.15;
  ENDDATA
  ! Minimize risk;
   [OBJ] MIN = @SUM( SCENE: PRB * ( DVL + DVU) ^ 2);
  !We are stuck with having asset 1 in the portfolio;
    X( 1) = 1;
  !Compute hedging portfolio value under each scenario;
    @FOR( SCENE( S):
      R( S)=
       @SUM( ASSET( J)| J #GT# 1: VE( S, J) * X( J));
  ! Measure deviations hedge + benchmark from target;
        DVU( S) - DVL( S) =
            ( R(S) + VE( S, 1))/ 2 - TARGET;
     );
  !  Budget constraint(applies to remaining assets);
   [BUDGET] @SUM( ASSET( J)| J #GT# 1: X( J)) = 1;
  ! Compute expected value of ending position;
   [DEFAVG] AVG = @SUM( SCENE: PRB * R);
  ! Target ending value;
   [RET] AVG > TARGET;
END
```

With a solution:

```
Optimal solution found at step:          4
Objective value:                 0.3441714E-01
Variable            Value         Reduced Cost
 X( GMT)         1.000000            0.0000000
 X( ATT)        0.5813256            0.0000000
 X( USX)        0.4186744            0.0000000
```

Notice we get the same portfolio as with the Markowitz model.

A scenario model for constructing a portfolio matching the S&P500 looks as follows:

```
MODEL:
 ! Scenario  model, Match 1st asset(PRTSMTCH);
 ! Minimize the variance;
  SETS:
   SCENE/1..12/: PRB, R, DVU, DVL;
   ASSET/ SP500  ATT  GMT  USX/:  X;
   SXA( SCENE, ASSET): VE;
  ENDSETS
 DATA:
 ! Data based on original Markowitz example;
  VE =
 !  S&P500   ATT    GMC    USX;
  1.258997    1.3  1.225  1.149
  1.197526  1.103  1.29   1.26
  1.364361  1.216  1.216  1.419
  0.919287  0.954  0.728  0.922
   1.05708  0.929  1.144  1.169
  1.055012  1.056  1.107  0.965
  1.187925  1.038  1.321  1.133
  1.31713   1.089  1.305  1.732
  1.240164  1.09   1.195  1.021
  1.183675  1.083  1.39   1.131
  0.990108  1.035  0.928  1.006
  1.526236  1.176  1.715  1.908;
 ! All scenarios happen to be equally likely;
 PRB= .0833333 .0833333 .0833333 .0833333 .0833333
      .0833333 .0833333 .0833333 .0833333 .0833333
      .0833333 .0833333;
 ! The desired return;
    TARGET = 1.191458;
 ENDDATA
 ! Minimize risk;
 [OBJ] MIN = @SUM( SCENE: PRB * ( DVL + DVU) ^ 2);
 ! Compute portfolio value under each scenario;
   @FOR( SCENE( S):
    R( S) =
    @SUM( ASSET( J)| J #GT# 1: VE( S, J) * X( J));
 ! Measure deviations of portfolio from benchmark;
       DVU( S) - DVL( S) = ( R(S) - VE( S, 1));
    );
 ! Budget constraint(applies to remaining assets);
  [BUDGET] @SUM( ASSET( J)| J #GT# 1: X( J)) = 1;
```

```
! Compute expected value of ending position;
 [DEFAVG] AVG = @SUM( SCENE: PRB * R);
! Target ending value;
 [RET] AVG > TARGET;
END
```

The solution is:

```
Optimal solution found at step:          7
Objective value:                0.4808974E-02
 Variable           Value        Reduced Cost
X( SP500)       0.0000000           0.0000000
  X( ATT)       0.2276583           0.0000000
  X( GMT)       0.4781151           0.0000000
  X( USX)       0.2942266           0.0000000
```

Notice we get the same portfolio as with the Markowitz model.

The two scenario models both used variance for the measure of risk relative to the benchmark. It is easy to modify them, so more asymmetric risk measures, such as downside risk, could be used.

13.8.2 Efficient Benchmark Portfolios

We say a portfolio is on the efficient frontier if there is no other portfolio that has both higher expected return and lower risk.

Let:

r_i = expected return on asset i,
t = an arbitrary target return for the portfolio.

A portfolio, with weight m_i on asset i, is efficient if there exists some target t for which the portfolio is a solution to the problem:

Minimize risk
subject to

$$\sum_{i=0}^{n} m_i = 1 \qquad \text{(budget constraint)}$$

$$\sum_{i=0}^{m} r_i\, m_i \geq t \qquad \text{(return target constraint).}$$

Portfolio managers are frequently evaluated on their performance relative to some benchmark portfolio. Let b_i = the weight on asset i in the benchmark portfolio. If the benchmark portfolio is not on the efficient frontier, then an interesting question is: What are the weights of the portfolio on the efficient frontier that is closest to the benchmark portfolio in the sense that the risk of the efficient portfolio relative to the benchmark is minimized?

There is a particularly simple answer when the measure of risk is portfolio variance, there is a risk-free asset, borrowing is allowed at the risk-free rate, and short sales are permitted. Let m_0 = the weight on the risk-free asset. An elegant result, in this case, is that there is a so-called "market" portfolio with weights m_i on asset i, such that effectively only m_0 varies as the return target varies. Specifically, there are constants m_i, for $i = 1, 2, \ldots, n$, such that the weight on asset i is simply $(1 - m_0) \times m_i$. Define:

$q = 1 - m_0$ = weight to put on the market portfolio,
R_i = random return on asset i.

Then the variance of any efficient portfolio relative to the benchmark portfolio can be written as:

$$\text{var}\left(\sum_{i=1}^{n} R_i[qm_i - b_i]\right)$$

$$= \sum_{i=1}^{n} (qm_i - b_i)^2 \text{ var}(R_i) + 2 \sum_{j>i} (qm_i - b_i)(qm_j - b_j) \text{ Cov}(R_i, R_j).$$

Setting the derivative of this expression with respect to q equal to zero gives the result:

$$q = \frac{\sum_{i=1}^{n} m_i b_i \text{ var}(R_i) + \sum_{j>i} (m_i b_j m_j b_i) \text{ Cov}(R_i, R_j)}{\sum_{i=1}^{n} m_i^2 \text{ var}(R_i) + 2 \sum_{j>i} m_i m_j \text{ Cov}(R_i, R_j)}$$

For example, if the benchmark portfolio is on the efficient frontier with weight b_0 on the risk-free asset, then $b_i = (1 - b_0)m_i$ and therefore $q = 1 - b_0$.

Thus, a manager who is told to outperform the benchmark portfolio $\{b_0, b_1, \ldots, b_n\}$ should perhaps, in fact, be compensated according to his performance relative to the efficient portfolio given by q above.

13.8.3 Efficient Formulation of Portfolio Problems

The amount of time it takes to solve a mathematical model may depend dramatically on how the model is formulated. This phenomenon is well known in integer programming circles. Below, we illustrate the same phenomenon for nonlinear programs. We give several different, but mathematically equivalent, formulations of a portfolio optimization model.

Formulation 1

$$\text{Minimize} \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij} x_i x_j$$

subject to

$$\sum_{j=1}^{n} x_j = 1$$

$$\sum_{j=1}^{n} r_j x_j = r_0$$

Formulation 2
We can exploit the fact that the covariance matrix is symmetric to rewrite the objective as:

$$\text{Min} \quad \sum_{i=1}^{n} x_i \left(q_{ii}\, x_i + 2 \sum_{j=i+1}^{n} q_{ij}\, x_j\right)$$

subject to

$$\sum_{j=1}^{n} x_j = 1$$

$$\sum_{j=1}^{n} r_j x_j = r_0$$

Formulation 3
We can separately compute the term multiplying x_i in the objective to get the formulation:

$$\text{Minimize} \sum_{i=1}^{n} x_i\, w_i$$

subject to
For each i;

$$w_i = q_{ii}\, x_i + 2 \sum_{j=i+1}^{n} q_{ij}\, x_j, \quad w_i \text{ a free variable}$$

$$\sum_{j=1}^{n} x_j = 1$$

$$\sum_{j=1}^{n} r_j x_j = r_0$$

We solved a specific instance of these formulations for a data set based on the performance of 19 stocks on the New York Stock exchange (IBM, Xerox, ATT, etc.). These models were solved as general nonlinear programs. The fact they were quadratic programs was not exploited.

The solution time in seconds for each formulation was:

Formulation	Time in seconds
1	2.16
2	1.5
3	0.82

Why the dramatic differences in solution time?

The advantage of formulation (2) over (1) is relatively obvious. Each function evaluation of the objective in (1) requires approximately $2 \times n \times n$ multiplications (2 multiplications for each of approximately $n \times n$ terms). For (2), the equivalent figure is about $n + n \times n/2$ multiplications.

Formulation (3) has essentially the same number of multiplications as (2). However, about $n \times n/2$ of them appear in linear constraints. The number of constraints has dramatically increased. However, these constraints are linear and the technology for efficiently processing linear constraints is well developed.

13.9 Cholesky Factorization for Quadratic Programs

There is another formulation comparable to formulation (3), but even more compact. Given a covariance matrix $\{q_{ij}\}$, one can compute its "square root," the so-called Cholesky factorization, to give a lower triangular matrix $\{L_{ij}\}$. The new formulation is then:

Minimize $\sum_{j=1}^{n} w_j^2$

subject to

For each j:

$w_j = \sum_{i=j+1}^{n} L_{ij}\, x_j$, w_j a free variable

$\sum_{j=1}^{n} x_j = 1$

$\sum_{j=1}^{n} r_j\, x_j = r_0$

Notice it is approximately identical in structure to formulation (3) except it has only n rather than $2n$ variables in the objective.

For the reader comfortable with matrix notation, the transformation is easy to explain. Given the covariance matrix Q and a lower triangular matrix L such that:

$L\,L' = Q$, where L' denotes transpose,

our objective is to:

Minimize $x\,Q\,x' = x\,L\,L'\,x'$

If we set $w = x\,L$, then our objective is simply:

Minimize $w\,w'$

subject to

$w = x\,L$.

A LINGO model using Cholesky decomposition and applied to our three-asset example is shown below:

```
MODEL:! Cholesky factorization Portfolio model;
SETS:
 ASSET: AMT, RET, CW;
 COVMAT( ASSET, ASSET): VARIANCE;
 MAT(ASSET,ASSET)| &1 #GE# &2: L; !Cholesky factor;
ENDSETS
```

```
DATA:
   ASSET =    ATT      GMC        USX;
!Covariance matrix and expected returns;
VARIANCE = .01080754 .01240721 .01307513
           .01240721 .05839170 .05542639
           .01307513 .05542639 .09422681;
     RET = .0890833  .213667   .234583;
ENDDATA
! Minimize variance;
[VAR] MIN = @SUM( ASSET( I): CW( I) * CW( I));
!  Use exactly 100% of the starting budget;
[BUDGET] @SUM( ASSET: AMT) = 1;
! Required wealth at end of period;
[RETURN] @SUM( ASSET: AMT * RET) > .15;
! Compute contributions to variance, CW();
@FOR( ASSET( J):
   @FREE( CW( J));
   CW( J) = @SUM( MAT( I, J): L( I, J) * AMT( I));
    );
!Compute the Cholesky factor L, so LL'= VARIANCE;
@FOR( ASSET( I):
 @FOR( MAT( I, J)| J #LT# I:
L(I,J) = ( VARIANCE( I, J) - @SUM( MAT( I, K)|
          K #LT# J: L( I, K) * L( J, K)))/ L( J, J);
          );
L(I,I) = ( VARIANCE( I, I) - @SUM( MAT( I, K)|
          K #LT# I: L( I, K) * L( I, K)))^.5;
          );
END
```

Part of the solution report is shown below:

```
Optimal solution found at step:          4
Objective value:                 0.2241375E-01
             Variable        Value    Reduced Cost
            AMT( ATT)    0.5300926       0.0000000
            AMT( GMC)    0.3564106       0.0000000
            AMT( USX)    0.1134968       0.4492217E-08
            RET( ATT)    0.8908330E-01   0.0000000
            RET( GMC)    0.2136670       0.0000000
            RET( USX)    0.2345830       0.0000000
             CW( ATT)    0.1119192       0.0000000
             CW( GMC)    0.9671834E-01   0.0000000
             CW( USX)    0.2309568E-01   0.0000000
VARIANCE( ATT, ATT)    0.1080754E-01   0.0000000
VARIANCE( ATT, GMC)    0.1240721E-01   0.0000000
VARIANCE( ATT, USX)    0.1307513E-01   0.0000000
VARIANCE( GMC, ATT)    0.1240721E-01   0.0000000
VARIANCE( GMC, GMC)    0.5839170E-01   0.0000000
VARIANCE( GMC, USX)    0.5542639E-01   0.0000000
VARIANCE( USX, ATT)    0.1307513E-01   0.0000000
VARIANCE( USX, GMC)    0.5542639E-01   0.0000000
VARIANCE( USX, USX)    0.9422681E-01   0.0000000
         L( ATT, ATT)    0.1039593       0.0000000
         L( GMC, ATT)    0.1193468       0.0000000
         L( GMC, GMC)    0.2101144       0.0000000
         L( USX, ATT)    0.1257716       0.0000000
         L( USX, GMC)    0.1923522       0.0000000
         L( USX, USX)    0.2034919       0.0000000
                Row   Slack or Surplus   Dual Price
                VAR      0.2241375E-01  -1.000000
             BUDGET      0.0000000       0.8255034E-02
             RETURN      0.0000000      -0.3538836
```

13.10 Positive Definiteness Constraints

An important feature of a valid covariance matrix is that it must be positive semi-definite. Loosely speaking, this means the diagonal of the matrix must be large relative to the off-diagonal elements. More precisely, if Q is a square matrix, e.g., a covariance matrix, then for any vector x, we must have, in matrix notation, $x'Qx \geq 0$. In terms of portfolio optimization, if the x vector represents the amount invested in a set of assets, and Q is the covariance matrix, then $x'Qx$ is the variance of the portfolio and we expect, and in fact require, that this variane be ≥ 0. Now suppose that the Q matrix is not given in advance, but rather the the elements of Q are decision variables, and we are constraining these elements of the matrix so that Q is positive semi-definite. A mathematical program in which we allow such constraints is called a Semi-Definite Program, or SDP for short. LINGO has a simple constraint type to indicate that a matrix Q must be positive semi-definite, namely, `@POSD( Q);`.

To illustrate the usefulness of this capability, suppose that we asked three experts to estimate the three covariances between three stocks and we obtained the following "guesstimate" of the correlation matrix:

```
1.000000   0.6938961 -0.1097276
0.6938961  1.000000   0.7972293
```

```
        -0.1097276  0.7972293  1.000000 ;
```

Although not immediately obvious, it happens to be the case that this matrix is not quite positive-semi-definte, so it is not a valid correlation matrix. So a reasonable thing to try to do is to make minimal adjustments to the off-diagonal elements to convert this matrix to a positive semi-definite matrix. The following LINGO model witll do this. Notice the following: 1) Because the matrix is symmetric, LINGO only requires that you enter the lower triangle of the matrix. 2) The last statement in the model is `@POSD( QFIT)`, i.e., we want the fitted matrix to be positive semi-definite, and 3) Our objective is to minimize the sum of the squared differences between the original guessed matrix and the fitted matrix.

```
SETS:
  VEC;
  MAT( VEC,VEC) | &1 #GE# &2: QINI, QADJ, QFIT;
ENDSETS
DATA:
  VEC = 1..3;
! Our initial estimate of the correlation matrix,
    ( May not be positive semi-definite);
  QINI =
         1.000000
         0.6938961  1.000000
        -0.1097276  0.7972293 1.000000 ;
ENDDATA

! Minimize the amount of adjustments we have
 to make to the off-diagonal terms of
 our initial estimated matrix...;
  MIN = @SUM( MAT(i,j) | i #GT# j: QADJ(i,j)^2);

! Fitted matrix = initial + adjustment;
  @FOR( MAT(i,j) | i #GT# j:
    QFIT(i,j) = QINI(i,j) + QADJ(i,j);
! Off diagonal adjustments or fitted
  might be  < 0;
  @FREE( QADJ(i,j));
  @FREE( QFIT(i,j));
       );

! Diagonal terms stay at 1;
 @FOR( VEC(i):
   QFIT(i,i) = QINI(i,i);
   QADJ(i,i) = 0;
     );

! The adusted/fitted matrix must be
  Positive semi-definite;
 @POSD( QFIT);
```

When solved, we get the fitted matrix:

```
        1.000000
```

```
 0.6348391       1.000000
-0.0640226       0.7304152       1.000000
```

Notice that in the fitted matrix, the off-diagonal elements have been moved closer to 0. There are a number of other applications of the @POSD( ) or SDP capability. Look at the MODELS library at www.lindo.com under the keyword of @POSD.

13.11 Problems

1. You are considering three stocks, *IBM*, *GM*, and Georgia-Pacific (*GP*), for your stock portfolio. The covariance matrix of the yearly percentage returns on these stocks is estimated to be:

	IBM	**GM**	**GP**
IBM	10	2.5	1
GM	2.5	4	1.5
GP	1	1.5	9

   Thus, if equal amounts were invested in each, the variance would be proportional to 10 + 4 + 9 + 2 (2.5 + 1 + 1.5). The predicted yearly percentage returns for *IBM*, *GM*, and *GP* are 9, 6 and 5, respectively. Find a minimum variance portfolio of these three stocks for which the yearly return is at least 7, at most 80% of the portfolio is invested in *IBM*, and at least 10% is invested in *GP*.

2. Modify your formulation of problem 1 to incorporate the fact that your current portfolio is 50% *IBM* and 50% *GP*. Further, transaction costs on a buy/sell transaction are 1% of the amount traded.

3. The manager of an investment fund hypothesizes that three different scenarios might characterize the economy one year hence. These scenarios are denoted *Green*, *Yellow* and *Red* and subjective probabilities 0.7, 0.1, and 0.2 are associated with them. The manager wishes to decide how a model portfolio should be allocated among stocks, bonds, real estate and gold in the face of these possible scenarios. His estimated returns in percent per year as a function of asset and scenario are given in the table below:

	Stocks	**Bonds**	**Real Estate**	**Gold**
Green	9	7	8	-2
Yellow	–1	5	10	12
Red	10	4	-1	15

   Formulate and solve the asset allocation problem of minimizing the variance in return subject to having an expected return of at least 6.5.

4. Consider the ATT/GMC/USX portfolio problem discussed earlier. The desired or target rate of return in the solved model was 15%.

   a) Suppose we desire a 16% rate of return. Using just the solution report, what can you predict about the standard deviation in portfolio return of the new portfolio?

b) We illustrated the situation where the opportunity to invest money risk-free at 5% per year becomes available. That is, this fourth option has zero variance and zero covariance. Now, suppose the risk-free rate is 4% per year rather than 5%. As before, there is no limit on how much can be invested at 4%. Based on only the solution report available for the original version of the problem (where the desired rate of return is 15% per year), discuss whether this new option is attractive when the desired return for the portfolio is 15%.

c) You have $100,000 to invest. What modifications would need to be made to the original ATT/GMC/USX model, so the answers in the solution report would come in the appropriate units (e.g., no multiplying of the numbers in the solution by 100,000)?

d) What is the estimated standard deviation in the value of your end-of-period portfolio in (c) if invested as the solution recommends?