
Optimization Modeling with LINGO

Sixth Edition

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Contents

Contents	iii
Preface	xiii
Acknowledgments.....	xiii
1	1
What Is Optimization?	1
1.1 Introduction.....	1
1.2 A Simple Product Mix Problem.....	1
1.2.1 Graphical Analysis	2
1.3 Linearity	5
1.4 Analysis of LP Solutions.....	6
1.5 Sensitivity Analysis, Reduced Costs, and Dual Prices.....	8
1.5.1 Reduced Costs	8
1.5.2 Dual Prices.....	8
1.6 Unbounded Formulations	9
1.7 Infeasible Formulations	10
1.8 Multiple Optimal Solutions and Degeneracy	11
1.8.1 The “Snake Eyes” Condition.....	13
1.8.2 Degeneracy and Redundant Constraints.....	15
1.9 Nonlinear Models and Global Optimization	16
1.10 Problems.....	18
2	21
Solving Math Programs with LINGO	21
2.1 Introduction.....	21
2.2 LINGO for Windows, Apple Mac, and Linux.....	21
2.2.1 LINGO Menu	23
2.2.2 Windows Menu	24
2.2.3 Help Menu.....	24
2.2.4 Summary.....	25
2.3 Getting Started on a Small Problem.....	25
2.4 Integer Programming with LINGO	26
2.4.1 Warning for Integer Programs	28
2.5 Solving an Optimization Model.....	28
2.6 Problems.....	29
3	31
Analyzing Solutions	31
3.1 Economic Analysis of Solution Reports.....	31
3.2 Economic Relationship Between Dual Prices and Reduced Costs.....	31
3.2.1 The Costing Out Operation: An Illustration	32
3.2.2 Dual Prices, LaGrange Multipliers, KKT Conditions, and Activity Costing	33
3.3 Range of Validity of Reduced Costs and Dual Prices	34
3.3.1 Predicting the Effect of Simultaneous Changes in Parameters—The 100% Rule	39
3.4 Sensitivity Analysis of the Constraint Coefficients.....	40
3.5 The Dual LP Problem, or the Landlord and the Renter	41

3.6 Problems.....	43
4.....	49
The Model Formulation Process.....	49
4.1 The Overall Process	49
4.2 Approaches to Model Formulation.....	50
4.3 The Template Approach.....	50
4.3.1 Product Mix Problems.....	50
4.3.2 Covering, Staffing, and Cutting Stock Problems.....	50
4.3.3 Blending Problems.....	50
4.3.4 Multiperiod Planning Problems.....	51
4.3.5 Network, Distribution, and PERT/CPM Models.....	51
4.3.6 Multiperiod Planning Problems with Random Elements.....	51
4.3.7 Financial Portfolio Models.....	51
4.3.8 Game Theory Models.....	52
4.4 Constructive Approach to Model Formulation.....	52
4.4.1 Example.....	53
4.4.2 Formulating Our Example Problem.....	53
4.5 Choosing Costs Correctly.....	54
4.5.1 Sunk vs. Variable Costs.....	54
4.5.2 Joint Products.....	56
4.5.3 Book Value vs. Market Value.....	57
4.6 Common Errors in Formulating Models.....	59
4.7 The Nonsimultaneity Error.....	61
4.8 Debugging a Model.....	61
4.9 Problems.....	63
5.....	67
The Sets View of the World.....	67
5.1 Introduction.....	67
5.1.1 Why Use Sets?.....	67
5.1.2 What Are Sets?.....	67
5.1.3 Types of Sets.....	68
5.2 The SETS Section of a Model.....	68
5.2.1 Defining Primitive Sets.....	68
5.2.2 Defining Derived Sets.....	69
5.2.3 Summary.....	70
5.3 The DATA Section.....	71
5.4 Set Looping Functions.....	73
5.4.1 @SUM Set Looping Function.....	74
5.4.2 @MIN and @MAX Set Looping Functions.....	75
5.4.3 @FOR Set Looping Function.....	76
5.4.4 Nested Set Looping Functions.....	77
5.5 Set Based Modeling Examples.....	77
5.5.1 Primitive Set Example.....	78
5.5.2 Dense Derived Set Example.....	81
5.5.3 Sparse Derived Set Example - Explicit List.....	83
5.5.4 A Sparse Derived Set Using a Membership Filter.....	88
5.6 Domain Functions for Variables.....	92

5.7 Spreadsheets and LINGO	92
5.8 Programming in LINGO	96
5.8.1 Building Blocks for Programming.....	96
5.8.2 Generating Graphs and Charts.....	98
5.9 Problems.....	100
6.....	101
Product Mix Problems	101
6.1 Introduction	101
6.2 Example.....	102
6.3 Process Selection Product Mix Problems	105
6.4 Problems.....	110
7.....	113
Covering, Staffing & Cutting Stock Models.....	113
7.1 Introduction	113
7.1.1 Staffing Problems.....	114
7.1.2 Example: Northeast Tollway Staffing Problems.....	114
7.1.3 Additional Staff Scheduling Features.....	116
7.2 Cutting Stock and Pattern Selection.....	117
7.2.1 Example: Cooldot Cutting Stock Problem.....	118
7.2.2 Formulation and Solution of Cooldot	119
7.2.3 Generalizations of the Cutting Stock Problem	123
7.2.4 Two-Dimensional Cutting Stock Problems	125
7.3 Crew Scheduling Problems	126
7.3.1 Example: Sayre-Priors Crew Scheduling.....	126
7.3.2 Solving the Sayre/Priors Crew Scheduling Problem	128
7.3.3 Additional Practical Details	131
7.4 A Generic Covering/Partitioning/Packing Model	132
7.5 Problems.....	134
8.....	145
Networks, Distribution and PERT/CPM.....	145
8.1 What's Special About Network Models.....	145
8.1.1 Special Cases	148
8.1.2 Fitting into Network Structure: Roads with No Left Turns.....	148
8.2 PERT/CPM Networks and LP.....	149
8.3 Activity-on-Arc vs. Activity-on-Node Network Diagrams.....	154
8.4 Crashing of Project Networks	155
8.4.1 The Cost and Value of Crashing.....	156
8.4.2 The Cost of Crashing an Activity	156
8.4.3 The Value of Crashing a Project.....	156
8.4.4 Formulation of the Crashing Problem	157
8.5 Resource Constraints in Project Scheduling	160
8.6 Path Formulations.....	162
8.6.1 Example	163
8.7 Path Formulations of Undirected Networks.....	164
8.7.1 Example	165
8.8 Double Entry Bookkeeping: A Network Model of the Firm	167
8.9 Extensions of Network LP Models.....	168

8.9.1 Multicommodity Network Flows	169
8.9.2 Reducing the Size of Multicommodity Problems	170
8.9.3 Multicommodity Flow Example	170
8.9.4 Fleet Routing and Assignment.....	173
8.9.5 Fleet Assignment	176
8.9.6 Leontief Flow Models	180
8.9.7 Activity/Resource Diagrams.....	184
8.9.8 Spanning Trees.....	186
8.9.9 Steiner Trees	188
8.10 Nonlinear Networks	192
8.11 Problems.....	195
9.....	203
Multi-period Planning Problems	203
9.1 Introduction	203
9.2 A Dynamic Production Problem.....	205
9.2.1 Formulation	205
9.2.2 Constraints.....	206
9.2.3 Representing Absolute Values.....	208
9.3 Multi-period Financial Models.....	209
9.3.1 Example: Cash Flow Matching	209
9.4 Financial Planning Models with Tax Considerations	213
9.4.1 Formulation and Solution of the WSDM Problem	214
9.4.2 Interpretation of the Dual Prices	215
9.5 Present Value vs. LP Analysis.....	216
9.6 Accounting for Income Taxes	217
9.7 Dynamic or Multi-period Networks.....	220
9.8 End Effects	222
9.8.1 Perishability/Shelf Life Constraints	223
9.8.2 Startup and Shutdown Costs	223
9.9 Non-optimality of Cyclic Solutions to Cyclic Problems	223
9.10 Problems.....	229
10.....	233
Blending of Input Materials	233
10.1 Introduction	233
10.2 The Structure of Blending Problems.....	234
10.2.1 Example: The Pittsburgh Steel Company Blending Problem	235
10.2.2 Formulation and Solution of the Pittsburgh Steel Blending Problem.....	236
10.3 A Blending Problem within a Product Mix Problem	238
10.3.1 Formulation	239
10.3.2 Representing Two-sided Constraints.....	240
10.4 Proper Choice of Alternate Interpretations of Quality Requirements	243
10.5 How to Compute Blended Quality	246
10.5.1 Example	246
10.5.2 Generalized Mean.....	247
10.6 Interpretation of Dual Prices for Blending Constraints	249
10.7 Fractional or Hyperbolic Programming	249
10.8 Multi-Level Blending: Pooling Problems.....	250

10.9 Problems.....	255
11.....	267
Formulating and Solving Integer Programs	267
11.1 Introduction.....	267
11.1.1 Types of Variables	267
11.2 Exploiting the IP Capability: Standard Applications.....	268
11.2.1 Binary Representation of General Integer Variables	268
11.2.2 Minimum Batch Size Constraints	268
11.2.3 Fixed Charge Problems	269
11.2.4 The Simple Plant Location Problem	269
11.2.5 The Capacitated Plant Location Problem (CPL).....	271
11.2.6 Modeling Alternatives with the Scenario Approach	273
11.2.7 Linearizing a Piecewise Linear Function, Discontinuous Case	274
11.2.8 Linearizing a Piecewise Linear Function, Continuous Case.....	276
11.2.9 An n Interval Piecewise Linear Function Using $\log(n)$ Binaries.....	279
11.2.10 Converting Multivariate Functions to Separable Functions	280
11.3 Outline of Integer Programming Methods	281
11.4 Computational Difficulty of Integer Programs	284
11.4.1 NP-Complete Problems	285
11.5 Problems with Naturally Integer Solutions and the Prayer Algorithm.....	286
11.5.1 Network LPs Revisited	286
11.5.2 Integral Leontief Constraints	286
11.5.3 Example: A One-Period MRP Problem.....	287
11.5.4 Transformations to Naturally Integer Formulations	289
11.6 The Assignment Problem and Related Sequencing and Routing Problems.....	291
11.6.1 Example: The Assignment Problem	291
11.6.2 The Traveling Salesperson Problem	293
11.6.3 Capacitated Multiple TSP/Vehicle Routing Problems.....	300
11.6.4 Minimum Spanning Tree.....	303
11.6.5 The Linear Ordering Problem	304
11.6.6 Quadratic Assignment Problem	306
11.7 Problems of Grouping, Matching, Covering, Partitioning, and Packing	310
11.7.1 Formulation as an Assignment Problem.....	311
11.7.2 Matching Problems, Groups of Size Two	311
11.7.3 Groups with More Than Two Members	314
11.7.4 Groups with a Variable Number of Members, Assignment Version	317
11.7.5 Groups with A Variable Number of Members, Packing Version	318
11.7.6 Groups with A Variable Number of Members, Cutting Stock Problem	321
11.7.7 Groups with A Variable Number of Members, Vehicle Routing.....	325
11.8 Linearizing Products of Variables	329
11.8.1 Example: Bundling of Products.....	330
11.9 Representing Logical Conditions.....	332
11.10 Problems.....	333
12.....	343
Decision making Under Uncertainty and Stochastic Programs	343
12.1 Introduction.....	343
12.1.1 Identifying Sources of Uncertainty.....	344

12.2 The Scenario Planning (SP) Approach	345
12.2.1 Formulation and Structure of an SP Problem	345
12.3 Single Stage Decisions Under Uncertainty	347
12.3.1 The News Vendor Problem	347
12.3.2 Multi-product Inventory with Repositioning	350
12.4 Multi-Stage Decisions Under Uncertainty	353
12.4.1 Stopping Rule and Option to Exercise Problems	354
12.4.2. An Option Exercise Stopping Problem	357
12.5 Expected Value of Perfect Information (EVPI)	358
12.6 Expected Value of Modeling Uncertainty	358
12.6.1 Certainty Equivalence	358
12.7 Risk Aversion	359
12.7.1 Downside Risk	360
12.7.2 Example	361
12.8 Dynamic Programming and Financial Option Models	364
12.8.1 Binomial Tree Models of Interest Rates	365
12.8.2 Binomial Tree Models of Foreign Exchange Rates	369
12.9 Decisions Under Uncertainty with an Infinite Number of Periods	371
12.9.1 Example: Cash Balance Management	373
12.10 Chance-Constrained Programs	376
12.11 Problems	377
13.....	379
Portfolio Optimization.....	379
13.1 Introduction	379
13.2 The Markowitz Mean/Variance Portfolio Model	379
13.2.1 Example	380
13.3 Dualing Objectives: Efficient Frontier and Parametric Analysis	383
13.3.1 Portfolios with a Risk-Free Asset	383
13.3.2 The Sharpe Ratio	386
13.4 Important Variations of the Portfolio Model	387
13.4.1 Portfolios with Transaction Costs	388
13.4.2 Example	388
13.4.3 Portfolios with Taxes	390
13.4.4 Factors Model for Simplifying the Covariance Structure	392
13.4.5 Example of the Factor Model	393
13.4.6 Scenario Model for Representing Uncertainty	394
13.4.7 Example: Scenario Model for Representing Uncertainty	395
13.5 Measures of Risk other than Variance	397
13.5.1 Value at Risk (VaR)	398
13.5.2 Example of VaR	398
13.5.3 VaR Anomalies	400
13.5.4 Conditional Value at Risk (CVaR)	401
13.6 Scenario Model and Minimizing Downside Risk	403
13.6.1 Semi-variance and Downside Risk	404
13.6.2 Downside Risk and MAD	406
13.6.3 Scenarios Based Directly Upon a Covariance Matrix	406
13.7 Hedging, Matching and Program Trading	408

13.7.1 Portfolio Hedging	408
13.7.2 Portfolio Matching, Tracking, and Program Trading	408
13.8 Methods for Constructing Benchmark Portfolios	409
13.8.1 Scenario Approach to Benchmark Portfolios	412
13.8.2 Efficient Benchmark Portfolios	414
13.8.3 Efficient Formulation of Portfolio Problems	415
13.9 Cholesky Factorization for Quadratic Programs	417
13.10 Positive Definiteness Constraints	419
13.11 Problems	420
14.....	423
Multiple Criteria and Goal Programming	423
14.1 Introduction	423
14.1.1 Alternate Optima and Multicriteria	424
14.2 Approaches to Multi-criteria Problems	424
14.2.1 Pareto Optimal Solutions and Multiple Criteria	424
14.2.2 Utility Function Approach	424
14.2.3 Trade-off Curves	425
14.2.4 Example: Ad Lib Marketing	425
14.3 Goal Programming and Soft Constraints	428
14.3.1 Example: Secondary Criterion to Choose Among Alternate Optima	429
14.3.2 Preemptive/Lexico Goal Programming	431
14.4 Minimizing the Maximum Hurt, or Unordered Lexico Minimization	434
14.4.1 Example	435
14.4.2 Finding a Unique Solution Minimizing the Maximum	435
14.5 Identifying Points on the Efficient Frontier	440
14.5.1 Efficient Points, More-is-Better Case	440
14.5.2 Efficient Points, Less-is-Better Case	442
14.5.3 Efficient Points, the Mixed Case	444
14.6 Comparing Performance with Data Envelopment Analysis	445
14.7 Problems	450
15.....	453
Economic Equilibria and Pricing	453
15.1 What is an Equilibrium?	453
15.2 A Simple Simultaneous Price/Production Decision	454
15.3 Representing Supply & Demand Curves in LPs	455
15.4 Auctions as Economic Equilibria	459
15.5 Multi-Product Pricing Problems	463
15.6 General Equilibrium Models of An Economy	467
15.7 Transportation Equilibria	469
15.7.1 User Equilibrium vs. Social Optimum	473
15.8 Equilibria in Networks as Optimization Problems	475
15.8.1 Equilibrium Network Flows	477
15.9 Problems	479
16.....	483
Game Theory and Cost Allocation	483
16.1 Introduction	483
16.2 Two-Person Games	483

16.2.1 The Minimax Strategy	484
16.3 Two-Person Non-Constant Sum Games	486
16.3.1 Prisoner's Dilemma	487
16.3.2 Choosing a Strategy	488
16.3.3 Bimatrix Games with Several Solutions	491
16.4 Nonconstant-Sum Games Involving Two or More Players	493
16.4.1 Shapley Value	495
16.5 The Stable Marriage/Assignment Problem	495
16.5.1 The Stable Room-mate Matching Problem	499
16.6 Should We Behave Non-Optimally to Obtain Information?	501
16.7 Problems	502
17.....	505
Inventory, Production, and Supply Chain Management	505
17.1 Introduction	505
17.2 One Period News Vendor Problem	505
17.2.1 Analysis of the Decision	506
17.3 Multi-Stage News Vendor	508
17.3.1 Ordering with a Backup Option	511
17.3.2 Safety Lotsize	513
17.3.3 Multiproduct Inventories with Substitution	514
17.4 Economic Order Quantity	518
17.5 The Q,r Model.....	519
17.5.1 Distribution of Lead Time Demand	519
17.5.2 Cost Analysis of Q,r	519
17.6 Base Stock Inventory Policy	524
17.6.1 Base Stock — Periodic Review	525
17.6.2 Policy.....	525
17.6.3 Analysis.....	525
17.6.4 Base Stock — Continuous Review	527
17.7 Multi-Echelon Base Stock, the METRIC Model.....	527
17.8 DC With Holdback Inventory/Capacity	531
17.9 Multiproduct, Constrained Dynamic Lot Size Problems	533
17.9.1 Input Data.....	534
17.9.2 Example	535
17.9.3 Extensions	540
17.10 Problems.....	541
18.....	543
Design & Implementation of Service and Queuing Systems	543
18.1 Introduction	543
18.2 Forecasting Demand for Services	543
18.3 Waiting Line or Queuing Theory.....	544
18.3.1 Arrival Process	545
18.3.2 Queue Discipline.....	546
18.3.3 Service Process	546
18.3.4 Performance Measures for Service Systems	546
18.3.5 Stationarity	547
18.3.6 A Handy Little Formula	547

18.3.7 Example	547
18.4 Solved Queuing Models	548
18.4.1 Number of Outbound WATS lines via Erlang Loss Model	549
18.4.2 Evaluating Service Centralization via the Erlang C Model	550
18.4.3 A Mixed Service/Inventory System via the M/G/ ∞ Model	551
18.4.4 Optimal Number of Repairmen via the Finite Source Model	552
18.4.5 Selection of a Processor Type via the M/G/1 Model	553
18.4.6 Multiple Server Systems with General Distribution, M/G/c & G/G/c	555
18.5 Critical Assumptions and Their Validity	557
18.6 Networks of Queues	557
18.7 Designer Queues	559
18.7.1 Example: Positive but Finite Waiting Space System	559
18.7.2 Constant Service Time. Infinite Source. No Limit on Line Length	562
18.7.3 Example Effect of Service Time Distribution	562
18.8 Problems	565
19.....	567
Design & Implementation of Optimization-Based Decision Support Systems	567
19.1 General Structure of the Modeling Process	567
19.1.1 Developing the Model: Detail and Maintenance	568
19.2 Verification and Validation	568
19.2.1 Appropriate Level of Detail and Validation.....	568
19.2.2 When Your Model & the RW Disagree, Bet on the RW	569
19.3 Separation of Data and System Structure	570
19.3.1 System Structure	570
19.4 Marketing the Model	571
19.4.1 Reports.....	571
19.4.2 Report Generation in LINGO	574
19.5 Reducing Model Size.....	576
19.5.1 Reduction by Aggregation.....	577
19.5.2 Reducing the Number of Nonzeroes	580
19.5.3 Reducing the Number of Nonzeroes in Covering Problems.....	580
19.6 On-the-Fly Column Generation	582
19.6.1 Example of Column Generation Applied to a Cutting Stock Problem	583
19.6.2 Column Generation and Integer Programming.....	589
19.6.3 Row Generation	590
19.7 Problems.....	591
References	593
INDEX	603

Preface

This book shows how to use the power of optimization, sometimes known as mathematical programming, to solve problems of business, industry, and government. The intended audience is students of business, managers, and engineers. The major technical skill required of the reader is to be comfortable with the idea of using a symbol to represent an unknown quantity.

This book is one of the most comprehensive expositions available on how to apply optimization models to important business and industrial problems. If you do not find your favorite business application explicitly listed in the table of contents, check the very comprehensive index at the back of the book.

There are essentially three kinds of chapters in the book:

1. introduction to modeling (chapters 1, 3, 4, and 19),
2. solving models with a computer (chapters 2, 5), and
3. application specific illustration of modeling with LINGO (chapters 6-18).

Readers completely new to optimization should read at least the first five chapters. Readers familiar with optimization, but unfamiliar with LINGO, should read at least chapters 2 and 5. Readers familiar with optimization and familiar with at least the concepts of a modeling language can probably skip to chapters 6-18. One can pick and choose from these chapters on applications. There is no strong sequential ordering among chapters 6-18, other than that the easier topics are in the earlier chapters. Among these application chapters, chapters 11 (on integer programming), and 12 (on stochastic programming) are worthy of special mention. They cover two computationally intensive techniques of fairly general applicability. As computers continue to grow more powerful, integer programming and stochastic programming will become even more valuable. Chapter 19 is a concluding chapter on implementing optimization models. It requires some familiarity with the details of models, as illustrated in the preceding chapters.

There is a natural progression of skills needed as technology develops. For optimization, it has been:

- 1) Ability to solve the models: 1950's
- 2) Ability to formulate optimization models: 1970's
- 3) Ability to use turnkey or template models: 1990's onward.

This book has no material on the mathematics of solving optimization models. For users who are discovering new applications, there is a substantial amount of material on the formulation of optimization models. For the modern "two minute" manager, there is a big collection of "off-the-shelf", ready-to-apply template models throughout the book.

Users familiar with the text *Optimization Modeling with LINDO* will notice much of the material in this current book is based on material in the LINDO book. The major differences are due to the two very important capabilities of LINGO: the ability to solve nonlinear models, and the availability of the set or vector notation for compactly representing large models.

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